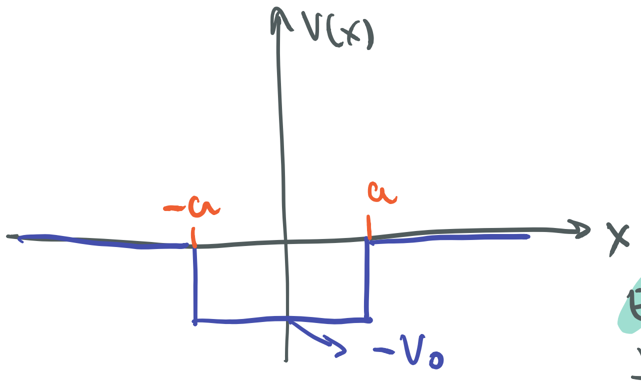
# Finite Square Well potential




$$\left[-\frac{\hbar^2}{2m}\partial_x^2 - V_0\right]\psi(x) = E\psi(x) \Big/ -a<x<a$$

$|x|<a$

$$-\frac{\hbar^2}{2m}\partial_x^2\psi(x) = E\psi(x)$$

Energy of the particle; it's the same of the two Eqs.

$E < V(\infty)$ : Bound state

$E > V(\infty)$ : Scattering state

As $V(\infty)=0 \Rightarrow$ bound states appear when $\underline{E<0}$

$$-\frac{\hbar^2}{2m}\partial_x^2\psi(x) = E\,\psi(x)$$

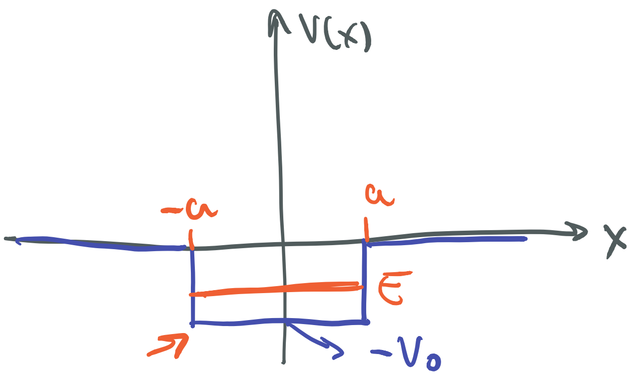
$$\partial_x^2\psi(x) = -\underline{\frac{2mE}{\hbar^2}}\,\psi(x)$$

$$-\frac{\hbar^2}{2m}\partial_x^2\psi(x) - V_0\psi(x) = E\psi(x)$$

$$\partial_x^2\psi(x) = -\underline{\frac{2m}{\hbar^2}(V_0+E)}\,\psi(x)$$

$$V_0+E>0$$

$$E>-V_0$$




$$\partial_x^2\psi(x) = K^2\psi(x)$$

$$K^2 = -\frac{2mE}{\hbar^2} > 0$$

$$\psi(x) = Ae^{-Kx} + Be^{Kx}$$

$$\partial_x^2\psi(x) = -\ell^2\psi(x)$$

$$\ell^2 = \frac{2m(V_0+E)}{\hbar^2} > 0$$

$$\psi(x) = C\sin(\ell x) + D\cos(\ell x)$$

The potential is symmetric!

$$V(x) = V(-x)$$

$$\psi(x) = \pm\psi(-x)$$

$$\psi(x): \begin{cases} \text{even} & \psi(x) = \psi(-x) \\ \text{odd} & \psi(x) = -\psi(-x) \end{cases}$$

$\cos(\ell x)$ is an even function

$\sin(\ell x)$ is an odd function

We will consider the even solutions, because the ground state is one these states.

Homework: Work out the odd solutions.

1. $\psi$ must be continuous
2. $\psi'$ must be continuous, except maybe where the potential diverges ($V(x) \to \infty$)

$V(x)$ never goes to $\infty$ $\Rightarrow$ $\psi'$ is continuous.

For $x>0$.

$$\psi(x) = A \underbrace{e^{-kx}}_{\text{vanishes}} + B \underbrace{e^{kx}}_{\text{diverges}} \qquad x \to \infty \qquad \Rightarrow B = 0$$

$$\psi(x) = D\cos(\ell x)$$

$$\psi(a) = \boxed{A e^{-ka} = D\cos(\ell a)}$$

For $x<0$:

$$\psi(x) = A \underbrace{e^{-kx}}_{\text{diverges}} + B \underbrace{e^{kx}}_{\text{vanishes}} \qquad x \to -\infty \qquad \Rightarrow \quad A = 0$$

$$\psi(x) = D\cos(\ell x)$$

$$\psi(-a) = B e^{-ka} = D\cos(-\ell a) \qquad \cos(x) = \cos(-x)$$

$$\boxed{B e^{-ka} = D\cos(\ell a)}$$

$$A e^{-ka} = D\cos(\ell a) = B e^{-ka} \quad \Rightarrow \quad \boxed{A = B}$$

Continuity of $\psi(x)$.

For $x>0$

$$\psi(x) = A e^{-kx} \Rightarrow \psi'(x) = -kA e^{-kx}$$

$\psi(x) = D\cos(lx) \Rightarrow \psi'(x) = -Dl\sin(lx)$

$-\kappa A e^{-\kappa a} = -Dl\sin(la)$

$\boxed{\kappa A e^{-\kappa a} = Dl\sin(la)}$

For $x<0$

$\psi(x) = Be^{\kappa x} \Rightarrow \psi'(x) = \kappa Be^{\kappa x}$

$\psi(x) = D\cos(lx) \Rightarrow \psi'(x) = -Dl\sin(lx)$

$\kappa B e^{-\kappa a} = -Dl\sin(-la)$ $\qquad \sin(-x) = -\sin(x)$

$\boxed{\kappa B e^{-\kappa a} = Dl\sin(la)}$

$Ae^{-\kappa a} = D\cos(la)$ $\qquad \kappa \underbrace{Ae^{-\kappa a}}_{Ae^{-\kappa a}} = l\underbrace{D\sin(la)}_{D\cos(la)}$

$\kappa = l\tan(la)$

And now?

$l = \dfrac{\sqrt{2m(V_0+E)}}{\hbar}$ $\qquad \kappa = \dfrac{\sqrt{-2mE}}{\hbar}$

$\kappa^2 + l^2 = \dfrac{-2mE}{\hbar^2} + \dfrac{2m}{\hbar^2}(V_0+E) = \dfrac{2mV_0}{\hbar^2}$

$z = la$ $\qquad l = \dfrac{z}{a}$ $\qquad \kappa = l\tan(la)$

$= \dfrac{z}{a}\tan(z)$

$\kappa^2 + l^2 = \left(\dfrac{z}{a}\right)^2\tan^2(z) + \left(\dfrac{z}{a}\right)^2 = \dfrac{2mV_0}{\hbar^2}$

$\left(\dfrac{z}{a}\right)^2\left[\tan^2(z)+1\right] = \dfrac{2mV_0}{\hbar^2}$

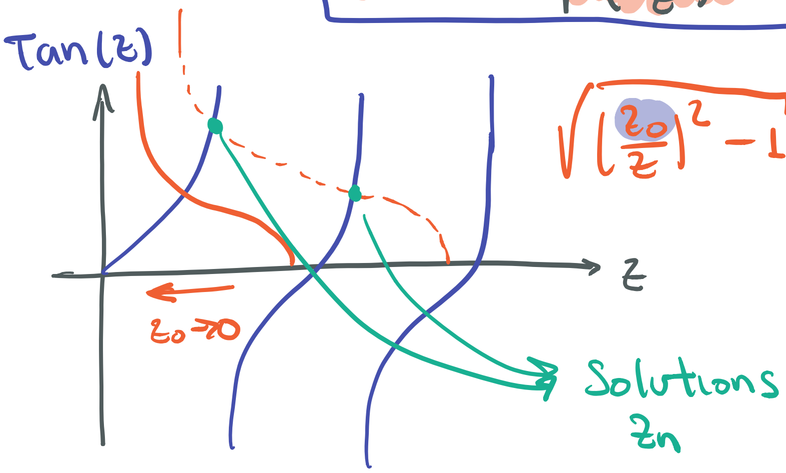
$\tan^2(z) + 1 = \dfrac{2mV_0}{\hbar^2}a^2\dfrac{1}{z^2}$

$= (z_0/z)^2$

Define $z_0^2 = \dfrac{2mV_0a^2}{\hbar^2}$

$$\tan^2(z) = \left(\frac{z_0}{z}\right)^2 - 1$$

$$\boxed{\tan(z) = \sqrt{\left(\frac{z_0}{z}\right)^2 - 1}} \leftarrow \quad z_n = la$$




$$z > z_0 \Rightarrow \left(\frac{z_0}{z}\right) < 1$$

$$\sqrt{\left(\frac{z_0}{z}\right)^2 - 1} \notin \mathbb{C}$$

No matter $V_0 < 1$ and $a < 1$, there's always at least one bound state

$$z_n = la$$

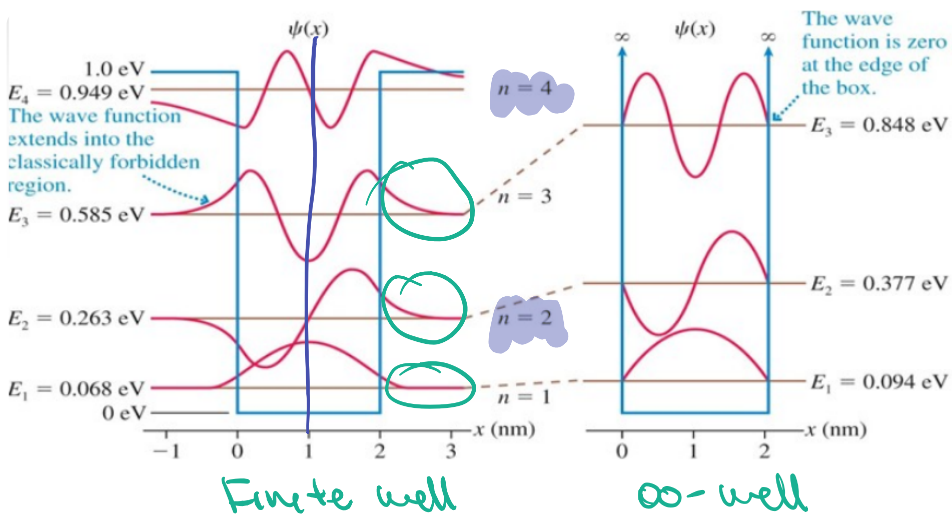
$$l = \frac{\sqrt{2m(V_0 + E)}}{\hbar} \Rightarrow l^2 = \frac{2m(V_0 + E)}{\hbar^2} \qquad l = \frac{z_n}{a}$$

$$E = \frac{\hbar^2 l^2}{2m} - V_0$$

$$= \left(\frac{\hbar z_n}{a}\right)^2 \frac{1}{2m} - V_0$$

When $V_0 \gg 1$, then $z_0 \gg 1$ $\quad z_n = \frac{n\pi}{2}$ for integer $n$

Recover the solutions for the $\infty$-well potential.




The energy diff. is smaller, because $\psi(x)$ can extend beyond the forbidden region.