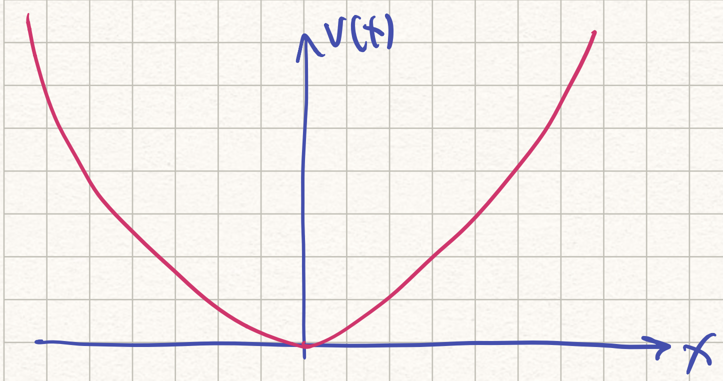
The Harmonic Oscillator

Potential is quadratic

$$V(x) = \frac{1}{2} m\omega^2 x^2$$



m: mass of the particle

ω: natural frequency of oscillation

$H\psi(x) = E\psi(x)$: Eigenvalue problem

$$H = \frac{p^2}{2m} + V(x) = \frac{(-i\hbar\partial_x)^2}{2m} + \frac{1}{2} m\omega^2 x^2$$

$$= \frac{1}{2m}\left[(-i\hbar\partial_x)^2 + m^2\omega^2 x^2\right]$$

$$\boxed{\frac{1}{2m}\left[(-i\hbar\partial_x)^2 + m^2\omega^2 x^2\right]\psi(x) = E\psi(x)}$$

$$u^2 + v^2 = (u+iv)(u-iv) \quad \begin{cases} u = -i\hbar\partial_x : \text{operator} \\ v = m\omega x \end{cases}$$

We define:

$$a_\pm \equiv \frac{i}{\sqrt{2m}}\left(-\hbar\partial_x \pm m\omega x\right)$$: these are also operators

$$a_+ a_- = \frac{i}{\sqrt{2m}}\left(-\hbar\partial_x + m\omega x\right)\frac{i}{\sqrt{2m}}\left(-\hbar\partial_x - m\omega x\right)$$

$$= \left(\frac{i}{\sqrt{2m}}\right)^2\left(\hbar^2\partial_x^2 + \hbar\partial_x(m\omega x) - m\omega x\hbar\partial_x - m^2\omega^2 x^2\right)$$

Operators only have physical meaning when we apply them on wavefunctions

$$a_+ a_- \psi(x) =$$

$$= \left(\frac{i}{\sqrt{2m}}\right)^2 \left(\hbar^2 \partial_x^2 + \hbar \partial_x (m\omega x) - m\omega x \hbar \partial_x - m^2\omega^2 x^2 \right) \psi(x)$$

$$= \frac{-1}{2m} \left(\hbar^2 \partial_x^2 \psi + \hbar \partial_x (m\omega x \psi) - m\omega x \hbar \partial_x \psi - m^2\omega^2 x^2 \psi \right)$$

$$= \frac{1}{2m} \Big(-\hbar^2 \partial_x^2 \psi + m^2\omega^2 x^2 \psi - \underbrace{\hbar \partial_x (m\omega x \psi)}_{\text{apply chain rule}} + m\omega x \hbar \partial_x \psi \Big)$$

$$\partial_x (m\omega x \psi) = m\omega \left(\partial_x (x) \psi + x \partial_x \psi \right)$$
$$= m\omega \psi + m\omega x \partial_x \psi$$

$$= \frac{1}{2m} \left(-\hbar^2 \partial_x^2 \psi + m^2\omega^2 x^2 \psi - \hbar (m\omega \psi + \cancel{m\omega x \partial_x \psi}) + \cancel{m\omega x \hbar \partial_x \psi} \right)$$

$$= \frac{1}{2m} \left(-\hbar^2 \partial_x^2 \psi + m^2\omega^2 x^2 \psi \right) - \frac{\hbar \omega m}{2m} \psi$$

$$= \frac{1}{2m} \left(-\hbar^2 \partial_x^2 + m^2\omega^2 x^2 \right) \psi - \frac{1}{2} \hbar\omega \psi$$

Because this is true for any WF ψ

$$a_+ a_- = \frac{1}{2m} \left(-\hbar^2 \partial_x^2 + m^2\omega^2 x^2 \right) - \frac{1}{2} \hbar\omega$$

$$= H - \frac{1}{2} \hbar\omega$$

The fact that ∂_x and x don't commute we have created a new term $-\frac{1}{2}\hbar\omega$

Follow these steps

$$a_- a_+ = \frac{1}{2m}\left(-\hbar^2 \partial_x^2 + m^2\omega^2 x^2\right) + \frac{1}{2}\hbar\omega$$

the extra $\hbar\omega$ is associated to the energy of the vaccum.

$$a_- a_+ - a_+ a_- = \frac{1}{2m}\cancel{\left(-\hbar^2 \partial_x^2 + m^2\omega^2 x^2\right)} + \frac{1}{2}\hbar\omega$$

$$-\left(\frac{1}{2m}\cancel{\left(-\hbar^2 \partial_x^2 + m^2\omega^2 x^2\right)} - \frac{1}{2}\hbar\omega\right)$$

$$= \hbar\omega$$

These operators don't commute

Commutator of two operators A and B:

$$[A, B] = AB - BA$$

$$[a_-, a_+] = \hbar\omega$$

Coming back to Schrödinger equation

If $a_+ a_- = H - \frac{1}{2}\hbar\omega \Rightarrow H = a_+ a_- + \frac{1}{2}\hbar\omega$

$$= a_- a_+ - \frac{1}{2}\hbar\omega$$

Our solution $\psi(x)$ should be such that

$$\left(a_+ a_- + \frac{1}{2}\hbar\omega\right)\psi = E\psi$$

$$\left(a_- a_+ - \frac{1}{2}\hbar\omega\right)\psi = E\psi$$

If $\psi(x)$ is a solution to S.E. with energy E then $\phi = a_+\psi$ is also a solution to S.E with energy $E + \hbar\omega$.

$$(a_+a_- + \tfrac{1}{2}\hbar\omega)\psi = E\psi$$

Let's see what happens with $\phi = a_+\psi$

$$\begin{aligned}(a_+a_- + \tfrac{1}{2}\hbar\omega)\,a_+\psi &= \left(a_+a_-a_+ + \tfrac{1}{2}\hbar\omega\, a_+\right)\psi \\ &= a_+\left(a_-a_+ + \tfrac{1}{2}\hbar\omega\right)\psi \\ &= a_+\left(a_-a_+ - \tfrac{1}{2}\hbar\omega + \hbar\omega\right)\psi \\ &= a_+\left\{\left(a_-a_+ - \tfrac{1}{2}\hbar\omega\right)\psi + \hbar\omega\psi\right\} \\ &= a_+\left\{E\psi + \hbar\omega\psi\right\} \\ &= a_+(E+\hbar\omega)\psi\end{aligned}$$

$$(a_+a_- + \tfrac{1}{2}\hbar\omega)\,a_+\psi = (E+\hbar\omega)\,a_+\psi$$

quanta

we have added a lump of energy

a_+ : commonly called the creation operator

Check that if ψ is a solution to S.E. with energy $E \Rightarrow a_-\psi$ is a solution to S.E. with energy $E - \hbar\omega$.

a_-: commonly called the annihilation operator

$\psi(x) \Rightarrow E$

$a_+\psi$	$\rightarrow E+\hbar\omega$	$a_-\psi$	$\rightarrow E-\hbar\omega$
$a_+(a_+\psi)$	$\rightarrow E+\hbar\omega+\hbar\omega$	$a_-(a_-\psi)$	$\Rightarrow E-\hbar\omega-\hbar\omega$
$a_+^2\psi$	$= E+2\hbar\omega$	$a_-^2\psi$	$= E-2\hbar\omega$
$\vdots$		$\vdots$	
$a_+^n\psi$	$\rightarrow E+n\hbar\omega$	$a_-^n\psi$	$\Rightarrow E-n\hbar\omega$