Reciprocal Space

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k)\, e^{i(kx-\omega t)}\, dk$$

Convention

up to a constant of Modulo 1

→ Fourier transform of $\psi(x,t)$

$\phi(k)$ contains all the information about the particle

$$\psi(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k)\, e^{ikx}\, dk$$

Fourier transform

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x',0)\, e^{-ikx'}\, dx'$$

Inverse Fourier transform

ψ and ϕ are "Fourier Pairs"

$$\begin{aligned}
\psi(x,0) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k)\, e^{ikx}\, dk \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, e^{ikx}\, \phi(k) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, e^{ikx}\, \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx'\, \psi(x',0)\, e^{-ikx'} \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx'\, \psi(x',0) \int_{-\infty}^{\infty} dk\, e^{ik(x-x')}
\end{aligned}$$

$2\pi\, \delta(x-x')$

$$\begin{aligned}
&= \int_{-\infty}^{\infty} dx'\, \psi(x',0)\, \delta(x-x') \\
&= \psi(x,0) \quad \blacksquare
\end{aligned}$$

$\psi(x,0) \rightarrow \phi(k)$

$\phi(k) \rightarrow \psi(x,0)$

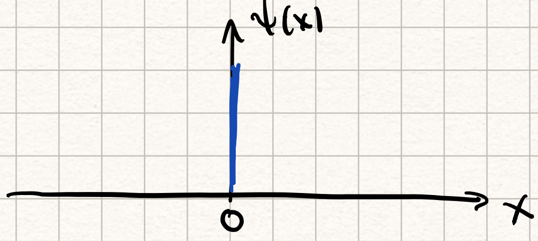
Particular Cases

1. $\phi(k) = 1$

$$\psi(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k)\, e^{ikx}\, dk$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 1\, e^{ikx}\, dk = \frac{1}{\sqrt{2\pi}}\, 2\pi\, \delta(x-0) \qquad 1 = e^{ik0}$$

$$= \sqrt{2\pi}\, \delta(x)$$



$\phi(k) = 1$ all the values of k have the same weight

we cannot say what "is" the k-value of the particle described by $\psi(x)$

we know that $\psi(x) = \sqrt{2\pi}\, \delta(x)$: localized at a single point!

we don't know the k-value the particle has, but we know precisely where the particle is!

2. $\psi(x) = \cos(x) = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$

We don't now where the particle is.

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-ikx}\, \psi(x,0)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{2}\, e^{-ikx} \left(e^{ix} + e^{-ix}\right) dx$$

$$= \frac{1}{2} \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} e^{-i(k-1)x}\, dx + \int_{-\infty}^{\infty} e^{-i(k+1)x}\, dx \right]$$

$$= \frac{1}{2} \frac{1}{\sqrt{2\pi}} \left[2\pi\, \delta(k-1) + 2\pi\, \delta(k+1) \right]$$

$$= \sqrt{2\pi}\, \frac{1}{2} \left[\delta(k-1) + \delta(k+1) \right]$$

$\phi(k)$ can only take $k=1$ or $k=-1$ but $\psi(x)$ is not localized

we know the k-values very well, but we don't know the position of the particle!

$\phi(k)$ and $\psi(x)$ have the same information

↳ same object in different spaces

$\psi(x)$ lives in the real space

- Positions (x) and time (t)

$\phi(k)$ lives in the reciprocal space

- Momenta (p) and frequencies (ω)

→ momentum space, k-space

$$\frac{\hat{p}^2}{2m}\psi(x) = \frac{\hbar^2 k^2}{2m}\psi(x) = E\,\psi(x) \qquad E = \frac{\hbar^2 k^2}{2m} = \frac{p^2}{2m}$$

$$\boxed{p = \hbar k}$$

speaking of k-values is equivalent to speaking of momenta!

Q.M. in reciprocal space

$$i\hbar\partial_t\psi(k,t) = \frac{p^2}{2m}\psi(k,t) = \frac{\hbar^2 k^2}{2m}\psi(k,t)$$

$$\hat{k}\psi(k,t) = k\,\psi(k,t)$$

$\psi(k,t)$ is eigenstate of $\hat{k}$ with eigenvalue k.

$$\langle\hat{k}\rangle = \int \psi^*(k)\,\hat{k}\psi(k)\,dk$$

$$= \int k\,\psi^*(k)\psi(k)\,dk$$

$$= \int k\,|\psi(k)|^2\,dk$$

Obtaining $\langle\hat{x}\rangle$ with $\psi(x,t)$

$\hat{p} \rightarrow -i\hbar\partial_x$

$\hat{x} \rightarrow i\hbar\partial_k$

Depending on the problem, it's more suitable to use one or the other spaces.

$\hat{x}$ and $\hat{k}$ are observables

t and ω are variables

Fourier idea also applies to variables!

$$\psi(x,t) \to \psi(x,\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x,t)\, e^{i\omega t}\, dt$$

$$\psi(k,\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x,\omega)\, e^{ikx}\, dx$$

$$\psi(k,\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{ikx}\, \psi(x,\omega)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{ikx} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, \psi(x,t)\, e^{i\omega t}$$

$$\psi(k,\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dt\, \psi(x,t)\, e^{ikx}\, e^{i\omega t}$$

Double Fourier transform

Wave function in the space of momentum (k) and energies ($\hbar\omega$)

Free particle with mass m : $E = \frac{\hbar^2 k^2}{2m} = \hbar\omega$

Dispersion relation

$$\omega(k) = \frac{\hbar k^2}{2m}$$

$\psi(k,\omega)$: complex function

Density of probability $\to$ $|\psi(k,\omega)|^2 = \psi^*(k,\omega)\,\psi(k,\omega)$

$S(k,\omega) = |\Psi(k,\omega)|^2$ Spectrum of the particle

Depending on the state of the particle, represent a cloud (set of points) occupying various parts of the dispersion relation.

Go to canvas: Reciprocal space

- Spectrum of a 1D particle occupying different states