Free Particle

$$H = \frac{p^2}{2m}$$

$$E_n = \frac{(\hbar k)^2}{2m} = \qquad k = \frac{n\pi}{L} \qquad \text{with } n \in \mathbb{Z}$$

$$= \frac{1}{2m}\left(\frac{n\pi\hbar}{L}\right)^2$$

$$\underbrace{E_{n+1}} - \underbrace{E_n} = \frac{1}{2m}\left[\frac{(n+1)\pi\hbar}{L}\right]^2 - \frac{1}{2m}\left[\frac{n\pi\hbar}{L}\right]^2$$

$$\Delta E = \frac{1}{L^2}\left\{\frac{1}{2m}\left[(n+1)\pi\hbar\right]^2 - \frac{1}{2m}(n\pi\hbar)^2\right]$$

What happens the length $L \to \infty$

$$\Delta E \propto \frac{1}{L^2} \quad \Rightarrow \quad L \to \infty \Rightarrow \Delta E \to 0$$

Classical limit

There's no quantization

Schrödinger Eq.

$$i\hbar \partial_t \Psi = H\Psi \qquad H = \frac{p^2}{2m} \qquad p \to -i\hbar\partial_x$$

$$= -\frac{\hbar^2}{2m}\partial_x^2 \Psi(x,t) \qquad = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$$

$$\Psi(x,t) = \psi(x)\, f(t)$$

Seperate the variables

$$i\hbar \frac{\partial_t f(t)}{f(t)} = E = -\frac{\hbar^2}{2m}\frac{\partial_x^2 \psi(x)}{\psi(x)}$$

$$\frac{1}{f}\frac{\partial}{\partial t} f(t) = -\frac{iE}{\hbar} \qquad -\frac{\hbar^2}{2m}\frac{\partial_x^2\psi(x)}{\psi(x)} = E$$

$$\int \frac{\partial f}{f} = -i\frac{E}{\hbar}\int \partial t \qquad \frac{\partial^2}{\partial x^2}\psi(x) = -\frac{2mE}{\hbar^2}\psi(x)$$

$$\rightarrow f(t) = f_0\, e^{-iEt/\hbar} \qquad = -k^2\psi(x) \qquad k^2 = \frac{2mE}{\hbar^2}$$

$\psi(x) = C \sin(kx) + D \cos(kx)$

Find the relation between coefficients A, B and C, D.

$\psi(x)$ can also be written as

$\psi(x) = A e^{ikx} + B e^{-ikx}$

$\Psi(x,t) = (A e^{ikx} + B e^{-ikx}) e^{-iEt/\hbar}$

f_0 is absorbed on A and B

$= A e^{i(kx - Et/\hbar)} + B e^{-i(kx + Et/\hbar)}$

very familiar from Optics
Two plane waves

propagating

towards the right

towards the left

Assumed $k > 0$ $\left(k^2 = \frac{2mE}{\hbar^2} \right)$

Let k take negative values:

Physical interpretation
→ direction of the propagation is reversed.

$$\boxed{\Psi(x,t) = A e^{i(kx - \omega t)}}$$

$\underline{\omega = E/\hbar}$

Check that ω has dimensions of a frequency!

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = |A|^2 \int_{-\infty}^{\infty} \underbrace{|e^{i(kx-\omega t)}|^2}_{1} dx = |A|^2 \int_{-\infty}^{\infty} dx$$

$$= 2|A|^2 \infty = \underline{\infty}$$

The free-particle is unphysical!

What happens if we take a sum over many frequencies (or k values)?

$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \Rightarrow \psi(x) = \sum_n a_n \psi_n(x)$

also a solution to the Schrödinger Eq.

$E_n = \frac{\hbar^2}{2m} \left(\frac{n\pi}{L}\right)^2$ $\qquad \sum_n \Leftrightarrow \sum_k$ over the allowed values of k.

For the free particle $E = \frac{\hbar^2 k^2}{2m}$ is not restrained

$$\sum_k \rightarrow \int dk$$

discrete set of values — continuous set of values

$$\Psi(x,t) = \int_{-\infty}^{\infty} \phi(k)\, e^{i(kx-\omega t)}\, dk$$

what happens with this state?

$\phi(k)$: the weight of each k-value into the $\Psi(x,t)$

$$\int_{-\infty}^{\infty} dx\, |\Psi(x,t)|^2 = \int_{-\infty}^{\infty} dx\, \Psi^*(x,t)\Psi(x,t)$$

$$\Psi^*(x,t) = \int_{-\infty}^{\infty} \phi^*(q)\, e^{-i(qx-\omega t)}\, dq$$

$$\Psi(x,t) = \int_{-\infty}^{\infty} dk\, \phi(k)\, e^{i(kx-\omega t)}$$

$$\Psi^*\Psi = \underbrace{\int_{-\infty}^{\infty} \phi^*(q)\, e^{-i(qx-\omega t)}\, dq}_{\text{indep. of } k} \underbrace{\int_{-\infty}^{\infty} dk\, \phi(k)\, e^{i(kx-\omega t)}}_{\text{indep. of } q}$$

$$= \int_{-\infty}^{\infty} dk\, \phi(k) \int_{-\infty}^{\infty} dq\, \phi^*(q)\, e^{i(kx-\omega t-qx+\omega t)}$$

$$= \int_{-\infty}^{\infty} dk\, \phi(k) \int_{-\infty}^{\infty} dq\, \phi^*(q)\, e^{i(k-q)x}$$

$$\int_{-\infty}^{\infty} dx\, |\Psi(x,t)|^2 = \int_{-\infty}^{\infty} dx\, \underbrace{\int_{-\infty}^{\infty} dk\, \phi(k) \int_{-\infty}^{\infty} dq\, \phi^*(q)}_{\text{ind. of } x}\, e^{i(k-q)x}$$

$$= \int_{-\infty}^{\infty} dk\, \phi(k) \int_{-\infty}^{\infty} dq\, \phi^*(q) \boxed{\int_{-\infty} dx\, e^{i(k-q)x}}$$

$\delta(k-q)$: Dirac delta function

$$\int_a^b f(x)\,\delta(x-x_0)\,dx = \begin{cases} f(x_0) & \text{if } a < x_0 < b \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} dx\, |\Psi(x,t)|^2 = \int_{-\infty}^{\infty} dk\, \phi(k) \int_{-\infty}^{\infty} dq\, \phi^*(q)\, \underbrace{\delta(k-q)}_{q=k}$$

$$= \int_{-\infty}^{\infty} dk\, \phi(k)\, \phi^*(k)$$

$$\int_{-\infty}^{\infty} dx\, |\Psi(x,t)|^2 = \boxed{\int_{-\infty}^{\infty} dk\, |\phi(k)|^2 = 1}$$

$$\rightarrow \Psi(x,t) = \int_{-\infty}^{\infty} \phi(k)\, e^{i(kx-\omega t)}\, dk$$

Is physical

All the results that we know from optics apply here!

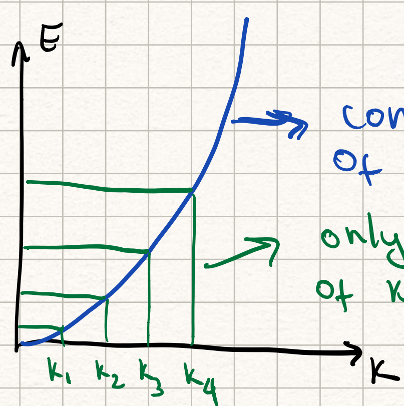
$$\boxed{E = \frac{\hbar^2 k^2}{2m}} \qquad E = \hbar\omega \qquad \hbar\omega = \frac{\hbar^2 k^2}{2m}$$

$$\omega(k) = \frac{\hbar k^2}{2m}$$

Dispersion relation

v_ϕ : phase velocity $\qquad v_\phi = \frac{\omega}{k} = \frac{\hbar k}{2m}$

v_g : group velocity $\qquad v_g = \frac{\partial \omega}{\partial k} = \frac{\hbar k}{m}$



→ continuum of values | free particle

→ only some values of k are allowed | particle in a box

$\phi(k)$: