Bosons: Ladder operators and Coherent states

<u>Bose</u> : Q. harmonic oscillator

- photons : light
- phonons : lattice vibrations
- excitons : electron + hole (electron vacancy)

$$\boxed{H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2}$$

m: mass of the particle

ω: associated frequency
- normal-mode frequency

$$a_\pm = \frac{i}{\sqrt{2m\hbar\omega}} \left(-\hbar\partial_x \pm m\omega x \right)$$: Ladder operators

$$\hat{a} = \frac{1}{\sqrt{2m\hbar\omega}} \left(\hat{p} - im\omega\hat{x} \right) \qquad \hat{a}^\dagger = \frac{1}{\sqrt{2m\hbar\omega}} \left(\hat{p} + im\omega\hat{x} \right)$$

$$\hat{p}^\dagger + im\omega\hat{x}^\dagger$$

$\hat{p}^\dagger = \hat{p}$, $\hat{x}^\dagger = \hat{x}$

$$p = -i\hbar\partial_x$$

$\hat{a}$: annihilation op.

$\hat{a}^\dagger$: creation op.

$$\hat{H}|\psi\rangle = E|\psi\rangle = \hbar\omega|\psi\rangle$$

$$|\psi'\rangle = a^\dagger|\psi\rangle \qquad \hat{H}|\psi'\rangle = (\hbar\omega + \hbar\omega)|\psi'\rangle = (E + \hbar\omega)|\psi'\rangle$$

$$|\phi'\rangle = a|\psi\rangle \qquad \hat{H}|\phi'\rangle = (E - \hbar\omega)|\phi'\rangle$$

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$$

<u>verify!</u>

$\{|0\rangle, |1\rangle, |2\rangle, \dots, |N\rangle, \dots\}$: Space of infinite dimension

countable discrete

not a maximum N_{max}

$\longrightarrow$ Fock basis

$$\langle n|m\rangle = \begin{cases} 1 & m=n \\ 0 & m\neq n \end{cases} = \delta_{n,m}$$

$$\underline{|\psi\rangle} = \sum_{n=0}^{\infty} c_n |n\rangle$$

$|n\rangle$: "Fock state n" state that contains n particles

$$\langle\psi| = \sum_{n=0}^{\infty} c_n^* \langle n|$$

$\underline{c_n} \in \mathbb{C} \Rightarrow \sum_n |c_n|^2 = 1$

a : removing 1 particle

$$a = \sum_{n=0}^{\infty} \sqrt{n}\, |n-1\rangle\langle n|$$

$$a^\dagger = \sum_{n=0}^{\infty} \sqrt{n+1}\, |n+1\rangle\langle n| \leftarrow$$

$$a\,\underline{|N\rangle} = \sum_{n=0}^{\infty} \sqrt{n}\, |n-1\rangle \underbrace{\langle n|N\rangle}_{\delta_{nN}}$$

$$= \underline{\sqrt{N}}\,\underline{|N-1\rangle}$$

$$a^\dagger\,\underline{|N\rangle} = \sum_{n=0}^{\infty} \sqrt{n+1}\, |n+1\rangle \underbrace{\langle n|N\rangle}_{\delta_{nN}}$$

$$= \sqrt{N+1}\,\underline{|N+1\rangle}$$

$$a^\dagger a = \left(\sum_k \sqrt{k+1}\,|k+1\rangle\langle k|\right)\left(\sum_n \sqrt{n}\,|n-1\rangle\langle n|\right)$$

$$= \sum_{k,n} \sqrt{n(k+1)}\, |k+1\rangle \underbrace{\langle k|n-1\rangle}_{\delta_{k=n-1}} \langle n|$$

$k+1 = (n-1)+1 = n$

$$= \sum_n \sqrt{n\cdot n}\, |n\rangle\langle n|$$

$$= \sum_n n\, |n\rangle\langle n|$$

$$H = \hbar\omega\left(\frac{1}{2} + \sum_n n\, |n\rangle\langle n|\right)$$

Hamiltonian in the Fock state basis.

$$|\psi\rangle = \frac{1}{\sqrt{1+|x|^2}}\,|0\rangle + \frac{x}{\sqrt{1+|x|^2}}\,|N\rangle \qquad x \in \mathbb{C}.$$

vacuum N-particles

$$\hat{H}|\psi\rangle = \frac{1}{\sqrt{1+|x|^2}}\,\hbar\omega\Big(\frac{1}{2} + \sum_n n\,\underbrace{|n\rangle\langle n|0\rangle}_{\delta_{n,0}}\Big)$$

$$+ \frac{x}{\sqrt{1+|x|^2}}\,\hbar\omega\Big(\frac{1}{2} + \sum_n n\,\underbrace{|n\rangle\langle n|N\rangle}_{\delta_{nN}}\Big)$$

$$= \frac{1}{\sqrt{1+|x|^2}}\,\hbar\omega\Big(\frac{1}{2} + 0\Big)|0\rangle + \frac{x}{\sqrt{1+|x|^2}}\,\hbar\omega\Big(\frac{1}{2} + N\Big)|N\rangle$$

$$= E\,|\psi\rangle$$

Find E that satisfies
Homework.

$$\langle\psi|\hat{H}|\psi\rangle = \Big(\frac{1}{\sqrt{1+|x|^2}}\langle 0| + \frac{x^*}{\sqrt{1+|x|^2}}\langle N|\Big)\,\hbar\omega\Big[\frac{1}{\sqrt{1+|x|^2}}\,\frac{1}{2}\,|0\rangle + \frac{x}{\sqrt{1+|x|^2}}\Big(\frac{1}{2}+N\Big)|N\rangle\Big]$$

$$= \hbar\omega\Big[\Big(\frac{1}{\sqrt{1+|x|^2}}\Big)^2\frac{1}{2} + \Big(\frac{x^*x}{\sqrt{1+|x|^2}}\Big)^2\Big(\frac{1}{2}+N\Big)\Big]$$

$$= \hbar\omega\ \frac{1}{2}\Big[\frac{1}{1+|x|^2} + \frac{|x|^2}{1+|x|^2}\Big] + \hbar\omega N\,\frac{|x|^2}{1+|x|^2}$$

$$= \frac{1}{2}\hbar\omega + N\hbar\omega\underbrace{\Big(\frac{|x|^2}{1+|x|^2}\Big)}$$

Vacuum energy
↓
Casimir effect

Coherent State : classical state

$$a\,|\alpha\rangle = \alpha\,|\alpha\rangle$$

α: $\mathbb{C}$; $|\alpha\rangle$: state

$$|\alpha\rangle = \sum_k c_k |k\rangle \quad , \quad c_k \in \mathbb{C}$$

$$a|\alpha\rangle = a \sum_k c_k |k\rangle$$

$$= \sum_k c_k \, a|k\rangle \qquad a|k\rangle = \underline{\underline{\sqrt{k}}} \, |k-1\rangle$$

$$= \sum_k c_k \sqrt{k} \, |k-1\rangle \quad \otimes$$

$$a|\alpha\rangle = \alpha |\alpha\rangle = \alpha \sum_k c_k |k\rangle = \sum_{k=0}^{\infty} c_k \alpha |k\rangle$$

$$\sum_{k=0}^{\infty} c_k \sqrt{k} \, |k-1\rangle = \sum_{k=1}^{\infty} c_k \sqrt{k} \, |k-1\rangle$$

$$q = k-1 \Rightarrow k = q+1$$

$$= \sum_{q=0}^{\infty} c_{q+1} \sqrt{q+1} \, |q\rangle$$

$$= \sum_{k=0}^{\infty} c_{k+1} \sqrt{k+1} \, |k\rangle = \sum_{k=0}^{\infty} c_k \alpha |k\rangle$$

$$\sum_{k=0}^{\infty} c_{k+1} \sqrt{k+1} \, |k\rangle - \sum_{k=0}^{\infty} c_k \alpha |k\rangle = 0$$

$$\sum_{k=0}^{\infty} \left(c_{k+1} \sqrt{k+1} - c_k \alpha \right) |k\rangle = 0$$

$$c_{k+1} \sqrt{k+1} - c_k \alpha = 0$$

$$\boxed{c_{k+1} = \frac{\alpha}{\sqrt{k+1}} c_k}$$

Recurrence relation

$$c_1 = \frac{\alpha}{\sqrt{1}} c_0$$

$$c_2 = \frac{\alpha}{\sqrt{2}} c_1 = \frac{\alpha}{\sqrt{2}} \cdot \frac{\alpha}{\sqrt{1}} c_0 = \frac{\alpha^2}{\sqrt{2 \cdot 1}} c_0$$

$$c_3 = \frac{\alpha}{\sqrt{3}} c_2 = \frac{\alpha^3}{\sqrt{3 \cdot 2 \cdot 1}} c_0$$

$$\vdots$$

$$c_n = \frac{\alpha^n}{\sqrt{n \cdot (n-1) \cdot (n-2) \cdots 1}} c_0 = \frac{\alpha^n}{\sqrt{n!}} c_0$$

$$n! = n \cdot (n-1)(n-2)(n-3) \cdots 1$$

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} c_0 |n\rangle = c_0 \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

↳ normalization of the state!

$$\langle\alpha|\alpha\rangle = 1 = \left(c_0^* \sum_{k=0}^{\infty} \frac{\alpha^{*k}}{\sqrt{k!}} \langle k|\right)\left(c_0 \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle\right) \rightarrow \text{simplify}$$

- Find the value of c_0 $\langle\alpha|\alpha\rangle = 1$.

$$a|\alpha\rangle = \alpha|\alpha\rangle$$

$$\langle\alpha|a^\dagger a|\alpha\rangle = (\langle\alpha|a^\dagger)(a|\alpha\rangle)$$
$$= \alpha^* \langle\alpha|\alpha|\alpha\rangle$$
$$= |\alpha|^2 \langle\alpha|\alpha\rangle$$
$$= |\alpha|^2$$: population of the coherent state.

$$(\langle\alpha|a^\dagger) = (a|\alpha\rangle)^\dagger$$
$$= (\alpha|\alpha\rangle)^\dagger$$
$$= \alpha^* \langle\alpha|$$

$\langle a^\dagger a\rangle$: mean number of particles populations

$\langle\psi|$ $|\psi\rangle$

$$|\psi'\rangle = a|\alpha\rangle = \alpha|\alpha\rangle$$

$$\langle\psi'|\psi'\rangle = 1 \Rightarrow |\psi'\rangle = |\alpha\rangle$$

- $\sigma_x \sigma_p = \hbar/2$: for $|\alpha\rangle$

$\sigma_x \sigma_p = \frac{\hbar}{2}(2N+1)$: for $|N\rangle$

Check this

Homework