Postulates of Quantum Mechanics

John von Neumann

1. The state of a quantum particle is represented by a normalized vector $|\psi\rangle$ in a Hilbert space

we can do algebra

Probabilistic nature of QM.

$x, p \rightarrow \psi(x) = \langle x|\psi\rangle$

$\psi(x)$: is familiar to us

$|\psi\rangle$: is the same vector in Dirac notation

2. Observables are represented by Hermitian operators

$\hat{O}$ correspond to measurements

$\psi(x)$ cannot be measured. $\rightarrow$ can be reconstructed from several measurements.

$\hat{O}$ act on a vector $\Rightarrow$ another vector in the same (Hilbert) space.

$|\psi\rangle$: $\hat{O}|\psi\rangle = |\phi\rangle$

$\hat{O}$ is Hermitian iff $\hat{O} = \hat{O}^{+}$

$\hat{x}$ and $\hat{p}$: they are linked to measurements

$\hat{p} = -i\hbar\partial_x \qquad \hat{Q}(\hat{x},\hat{p}) \Rightarrow Q(x, -i\hbar\partial_x)$

Example: kinetic energy

$\frac{\hat{p}^2}{2m} = \frac{-\hbar^2\partial_x^2}{2m} \Rightarrow$ the measurement is $\frac{p^2}{2m}$

3. The possible measurements of $\hat{Q}$ are its eigenvalues and the state after the measurement is the corresponding eigenvector (Collapse of the WF).

$$\hat{Q}|\lambda_i\rangle = \underset{\text{scalar}}{\lambda_i}\ \underset{\text{vector}}{|\lambda_i\rangle}$$

As many $|\lambda_i\rangle$ as $\dim(\hat{Q})$

$|\lambda_i\rangle$ for $i = 1, 2, \dots, n$

↳ orthonormalised

$$\langle\lambda_i|\lambda_j\rangle = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$$

Then $|\lambda_i\rangle$ are a basis for the space that contains $|\lambda\rangle$

$$|\psi\rangle = \sum_{i=1}^{n} \alpha_i |\lambda_i\rangle \qquad \alpha_i \in \mathbb{C}$$

$\sum_{k=1}^{n} |\lambda_k\rangle\langle\lambda_k|$: sum of projections

$$\left(\sum_{k=1}^{n} |\lambda_k\rangle\langle\lambda_k|\right)|\psi\rangle = \left(\sum_{k=1}^{n} |\lambda_k\rangle\langle\lambda_k|\right)\sum_{i=1}^{n} \alpha_i |\lambda_i\rangle$$

$$= \sum_{k,i=1}^{n} |\lambda_k\rangle\langle\lambda_k| \alpha_i |\lambda_i\rangle$$

$$= \sum_{k,i=1}^{n} \alpha_i |\lambda_k\rangle \underbrace{\langle\lambda_k|\lambda_i\rangle}_{\delta_{ki}}$$

$$= \sum_{k,i=1}^{n} \alpha_i |\lambda_k\rangle \delta_{ki}$$

$$= \sum_{i=1}^{n} \alpha_i |\lambda_i\rangle$$

$$= |\psi\rangle$$

$\sum_k |\lambda_k\rangle\langle\lambda_k| = \mathbb{1}$ The unit operator!

For the continuous Hilbert space $\mathbb{1} = \int dx\, |x\rangle\langle x|$

4. Born rule: If a particle is in the general state $|\psi\rangle = \sum_{i=1}^{n} \alpha_i |\lambda_i\rangle$, with $|\lambda_i\rangle$ the eigenvalues of $\hat{Q}$, the probability to get the particular eigenvalue λ_i when measuring $\hat{Q}$ is $|\alpha_i|^2$.
The state then collapses to $|\lambda_i\rangle$ as a result of the measurement.

Linear algebra: the eigenvectors of a Hermitian operator span the finite-dimensional Hilbert space

Any vector $|u\rangle$ can be written as a linear superposition of these eigenvectors

↳ these eigenvectors are a basis for the Hilbert space

$$|u\rangle = \sum_{i=1}^{n} \alpha_i |\lambda_i\rangle \qquad \text{with } \alpha_i \in \mathbb{C}$$

$$P(\lambda_k) = |\langle \lambda_k | u \rangle|^2$$ Projection of $|u\rangle$ onto $|\lambda_k\rangle$.

$$\langle \lambda_k | u \rangle = \langle \lambda_k | \sum_{i=1}^{n} \alpha_i |\lambda_i\rangle = \sum_{i=1}^{n} \alpha_i \underbrace{\langle \lambda_k | \lambda_i \rangle}_{\delta_{ki}}$$

$$= \alpha_k$$

$$|\langle \lambda_k | u \rangle|^2 = |\alpha_k|^2$$ Prob. to measure λ_k is $|\alpha_k|^2$.

$$(|\lambda_k\rangle\langle\lambda_k|)|u\rangle = |\lambda_k\rangle\langle\lambda_k| \sum_{i=1}^{n} \alpha_i |\lambda_i\rangle$$

$$= \sum_{i=1}^{n} \alpha_i |\lambda_k\rangle \underbrace{\langle \lambda_k | \lambda_i \rangle}_{\delta_{ki}}$$

$$|\lambda_k\rangle \langle \lambda_k | u \rangle = \alpha_k |\lambda_k\rangle$$

$\alpha_k = \langle \lambda_k | u \rangle$

If we start with $|u\rangle$

↳ measure $\hat{Q}$ and find λ_i

↳ the state vector is now $|\lambda_i\rangle$ The WF has collapsed to $|\lambda_i\rangle$!

Example: Quantum Coin

$$|\psi\rangle = \alpha |H\rangle + \beta |T\rangle \qquad |\alpha|^2 + |\beta|^2 = 1$$

- Prob. to get $|H\rangle$ is $|\alpha|^2$
  $|T\rangle$ is $|\beta|^2$

If we measure and find heads $\Rightarrow |\psi\rangle = |H\rangle$

- Prob. to get $|H\rangle = 1$
  $|T\rangle = 0$

The WF has collapsed
= It's no longer a superposition

Operator $\hat{M}$ with eigenvector $|\mu_i\rangle$ (as many as $|\lambda_i\rangle$)

$$|u\rangle = \sum_{i=1}^{n} \alpha_i |\lambda_i\rangle = \sum_{i=1}^{n} \beta_i |\mu_i\rangle \qquad \beta_i \in \mathbb{C}$$

$$\beta_i = \langle \mu_i | u \rangle \qquad |\mu_i\rangle = \sum_{j=1}^{n} \mu_{i,j} |\lambda_i\rangle$$

$$|\lambda_i\rangle = \sum_{j=1}^{n} \lambda_{i,j} |\mu_i\rangle$$

- start with $|u\rangle = \sum_{i=1}^{n} \alpha_i |\lambda_i\rangle$

↳ measure $\hat{Q}$ and find $|\lambda_j\rangle$ $\quad |\langle \lambda_j | u \rangle|^2$

↳ the state collapses to $|\lambda_j\rangle$ $\quad \times$

↳ measure $\hat{M}$ and find $|\mu_k\rangle$ $\quad |\langle \mu_k | \lambda_j \rangle|^2$

↳ the state collapses to $|\mu_k\rangle$ $\quad \times$ $\quad |\langle \lambda_n | \mu_k \rangle|^2$

↳ what's the prob. to measure $\hat{Q}$ and find $|\lambda_n\rangle$?

Write this prob. and send by email!

5. The time evolution of the state vector is given by Schrödinger Eq. $i\hbar\partial_t|\psi\rangle = \hat{H}|\psi\rangle$

- Separation of variables ⇒ time-ind. Schrödinger Eq.

$$\hat{H}|\psi\rangle = E|\psi\rangle$$: eigenvalue problem for the Hamiltonian (energy) operator