Vector spaces and linear algebra

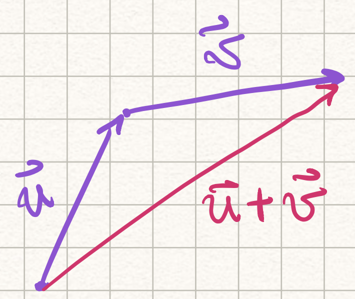
John von Neumann → unification of Q.M. maths
↳ Mathematical rules of Q.M.

Vector space

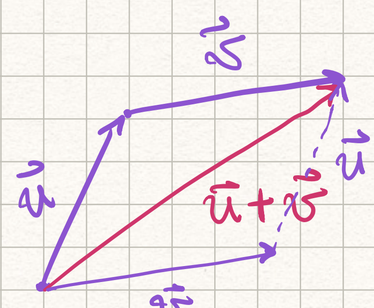
- Algebraic structure

→ we are used to geometrical vector

$\vec{u}, \vec{v}\ ; \Rightarrow \vec{v} + \vec{u}$



Tail-to-head method



Parallelogram method

In quantum physics we use the Dirac notation

$|\ldots\rangle$: vector : "a ket"

"..." what labels our vector

A vector space consists of a set of vectors

$|\alpha\rangle, |\beta\rangle, |\gamma\rangle, \ldots$

and some scalars ("numbers"), $a, b, c \ldots \in \mathbb{C}$

$a, b, c \ldots \in \mathbb{C}$: $a, b, c, \ldots$ belong to the set of complex numbers

Algebra → vector addition:

$|\alpha\rangle + |\beta\rangle = |\kappa\rangle$ is also in the vector space

scalar multiplication

$a|\alpha\rangle = |\kappa\rangle$ is also in the vector space

Linear combination

$$a|\alpha\rangle + b|\beta\rangle + c|\gamma\rangle + \cdots$$

$|\lambda\rangle$ is linearly independent from the set $|\alpha\rangle, |\beta\rangle, |\gamma\rangle \ldots$

If we cannot write

$$|\lambda\rangle \neq a|\alpha\rangle + b|\beta\rangle + c|\gamma\rangle \cdots$$

A set of vectors is linearly independent if no vector in the set is a linear combination of the others

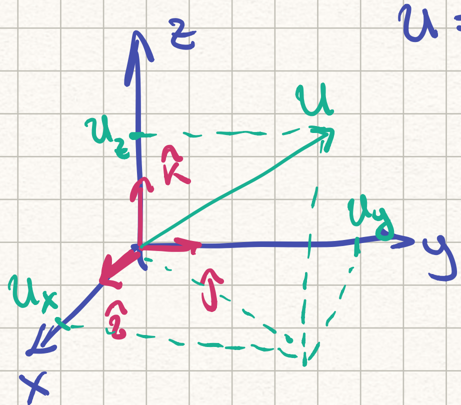
A set of vectors spans the vector space if every vector (in the space) can be written as a linear superposition of the vectors in the set

A set of linearly independent vectors that spans the space is a basis

The number of vectors in a basis is the dimension of the space

$\boxed{\text{Example:}}$

In 3D



$$\vec{u} = u_x \hat{\imath} + u_y \hat{\jmath} + u_z \hat{k}$$

with $u_x, u_y, u_z \in \mathbb{C}$.

$\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ is the basis of the 3D cartesian space, which has dimension 3

Finite-dimensional space

All vectors can be written as a finite sum of other vectors (on the basis)

Basis: $\{ |e_1\rangle, |e_2\rangle, \dots, |e_n\rangle \}$

dimension is n

Every vector $|\alpha\rangle$ can be written as

$$|\alpha\rangle = a_1 |e_1\rangle + a_2 |e_2\rangle + \dots + a_n |e_n\rangle$$

The scalars $a_1, \dots a_n$ are the components or coordinates of $|\alpha\rangle$ in the basis of $|e_i\rangle$

$$|\alpha\rangle = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

A vector can be seen as a column of numbers

$$\vec{u} = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}$$

These are a particular case of the algebra of Tables

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix}$$

- Two-index object

Matrix (pl. matrices)

rectangular $(m \neq n)$ or squared $(m = n)$ matrix

Matrices also follow their algebra of addition (adding components one by one) and scalar multiplication (rescalling all elements)

Operation on Matrices

Transposition

The transpose (A^T: "A transposed") interchanges rows and columns

$$A^T = \begin{pmatrix} a_{11} & a_{21} & a_{31} & \cdots & a_{m1} \\ a_{12} & a_{22} & a_{32} & \cdots & a_{m2} \\ a_{13} & a_{23} & a_{33} & \cdots & a_{m3} \\ \vdots & & & & \\ a_{1n} & a_{2n} & a_{3n} & \cdots & a_{mn} \end{pmatrix}$$

A $m \times n$ matrix yields A^T which is $n \times m$

If A is squared ($m=n$), the determinant

$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{vmatrix}$$

Example:

$$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Note that $|\alpha\rangle^T$

$$|\alpha\rangle = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \Rightarrow |\alpha\rangle^T = (a_1, a_2, \ldots, a_n)$$

when working with complex-valued vector fields, it's better to also take the complex conjugate when transposing

Apply the "transpose-conjugate" operation

- Hermitian conjugate => Hermite
- Adjoint

We use the symbol $A^{\dagger}$: "A dagger"

$A^{\dagger} \equiv (A^{T})^{*}$

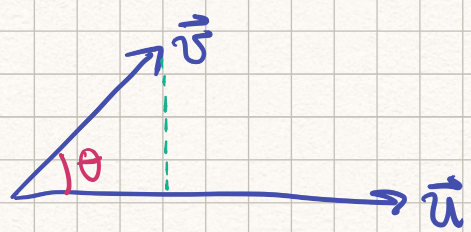
$$|\alpha\rangle^{\dagger} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}^{T*} = (a_1^*, a_2^*, \ldots, a_n^*) \equiv \langle\alpha|$$

"The bra of $|\alpha\rangle$"

$$|\alpha\rangle^{\dagger} \equiv \langle\alpha|$$

Scalar product

$\vec{u}\cdot\vec{v} = uv\cos\theta$: Shadow that $\vec{v}$ casts upon $\vec{u}$.



$$\vec{u}\cdot\vec{v} = u_x v_x + u_y v_y + u_z v_z$$

For abstract vectors the scalar product is called inner product

$$\langle\beta|\alpha\rangle : \text{scalar}$$

$\langle\beta| = (b_1^*, b_2^*, \ldots b_n^*)$

$$\langle\beta|\alpha\rangle = (b_1^*, b_2^*, \ldots b_n^*)\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

$$= b_1^* a_1 + b_2^* a_2 + \cdots + b_n^* a_n \ : \text{scalar}$$

If we take

$$\langle\alpha|\alpha\rangle = a_1^* a_1 + a_2^* a_2 + \cdots + a_n^* a_n$$
$$= |a_1|^2 + |a_2|^2 + \cdots + |a_n|^2 \quad : \text{Real number}$$

The norm (lenght) of the vector:

$$\|\alpha\| \equiv \sqrt{\langle\alpha|\alpha\rangle}$$

Two vectors that have zero scalar product are orthogonal

Vectors in a basis should

- be normalized: $\|e_i\| = 1$
- be mutually orthogonal: $\langle e_i|e_j\rangle = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$

δ_{ij} : Kronecker symbol

Applications or transformations

Functions that act on vectors

Let's assume a linear function T

$$T\left(a|\alpha\rangle + b|\beta\rangle\right) = aT|\alpha\rangle + bT|\beta\rangle$$

we have full knowledge of T if we know the effect that T has on every element of the basis

$$T|e_1\rangle = T_{11}|e_1\rangle + T_{21}|e_2\rangle + \cdots + T_{n1}|e_n\rangle$$
$$T|e_2\rangle = T_{12}|e_1\rangle + T_{22}|e_2\rangle + \cdots + T_{n2}|e_n\rangle$$
$$\vdots$$
$$T|e_n\rangle = T_{1n}|e_1\rangle + T_{2n}|e_2\rangle + \cdots + T_{nn}|e_n\rangle$$

For any vector $|\alpha\rangle$ we compute $T|\alpha\rangle$ by decomposing $|\alpha\rangle$ in the elem. of the basis

$$|\alpha\rangle = \sum_{i=1}^{n} a_i |e_i\rangle = a_1|e_1\rangle + a_2|e_2\rangle + \cdots + a_n|e_n\rangle$$

$$T|\alpha\rangle = \sum_{i=1}^{n} a_i T|e_i\rangle$$

$$T|e_i\rangle = T_{1i}|e_1\rangle + T_{2i}|e_2\rangle + \cdots + T_{ni}|e_n\rangle$$

$$= \sum_{k=1}^{n} T_{ki}|e_k\rangle$$

$$T|\alpha\rangle = \sum_{i=1}^{n} a_i \sum_{k=1}^{n} T_{ki}|e_k\rangle$$

$$= \sum_{k=1}^{n}\left(\sum_{i=1}^{n} T_{ki}a_i\right)|e_k\rangle = \begin{pmatrix} \sum_{i=1}^{n} T_{1i}a_i \\ \sum_{i=1}^{n} T_{2i}a_i \\ \vdots \\ \sum_{i=1}^{n} T_{ni}a_i \end{pmatrix} = T|\alpha\rangle$$

The transformation is reduced to a matrix

$$T = \begin{pmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{pmatrix}$$

Hermitian transformation

$T = T^{\dagger}$: T is Hermitian

Unitary : $T^{-1} = T^{\dagger}$: T is unitary

Eigenvector

Don't get rotated in space, only rescaled

$T|\alpha\rangle = \lambda|\alpha\rangle$: with λ a scalar

$|\alpha\rangle$ is an eigenvector of T with eigenvalue λ

$$T|\alpha\rangle = \lambda\mathbb{1}|\alpha\rangle$$

$|\alpha\rangle$ is nonzero

$$T|\alpha\rangle - \lambda\mathbb{1}|\alpha\rangle = 0$$

$(T-\lambda\mathbb{1})|\alpha\rangle = 0 \Rightarrow T-\lambda\mathbb{1}$ has to be singular
not invertible

$$\det(T-\lambda\mathbb{1}) = \begin{vmatrix} T_{11}-\lambda & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22}-\lambda & \cdots & T_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn}-\lambda \end{vmatrix} = 0$$

polynomial equation for λ of order n.

For Hermitian matrices

- Eigenvalues of a Hermitian transformation are real number
- Eigenvectors of a Hermitian transformation belonging to distinct eigenvalues are orthogonal
- The eigenvector of a Hermitial transformation span the space

Infinite-dimensional spaces

- Countable (like integers)
- Continuous (like the set of real numbers)

vectors can be functions $\Rightarrow$ function spaces

Upgrade

$$\sum \longrightarrow \int$$

We could write the scalar product in the following way (between two vector functions)

$$\langle f | g \rangle = \int f^*(x)\, g(x)\, dx$$

Transformations will be done through linear operator. For instance

$$\frac{d}{dx}, \quad \frac{d^2}{dx^2}, \quad \text{or simply: } x$$