Momentum in quantum mechanics

$$i\hbar\, \partial_t \psi(x,t) = -\frac{\hbar^2}{2m} \partial_x^2 \psi(x,t) + V(x,t)\,\psi(x,t)$$

- Position : where the particle is
- Momentum: where is the particle going and how fast

In classical physics

$$\vec{p} = m\vec{v} \qquad \vec{v} = \frac{\partial}{\partial t}\vec{x}$$

$$= m\frac{\partial}{\partial t}\vec{x}$$

$$= \frac{\partial}{\partial t}(m\vec{x})$$

In quantum physics

$$\langle p \rangle \equiv m\frac{\partial}{\partial t}\langle x \rangle$$

Because classical physics is a limiting case of quantum physics, we expect this relation to be true.

$$\langle p \rangle = m\frac{\partial}{\partial t}\int x\, \psi^*\psi\, dx = m\frac{\partial}{\partial t}\int x\, |\psi|^2 dx$$

$$= m\int \frac{\partial}{\partial t}(x|\psi|^2)\, dx$$

$$= m\int x\, \frac{\partial}{\partial t}|\psi|^2\, dx$$

Obtain $\partial_t |\psi|^2$

$$\partial_t |\psi|^2 = \partial_t(\psi^*\psi) = (\partial_t\psi^*)\psi + \psi^*(\partial_t\psi)$$ chain rule

$$\partial_t \psi = \frac{1}{i\hbar}\left[\frac{-\hbar^2}{2m}\partial_x^2\psi + V\psi\right]$$

$$\partial_t \psi^* = \frac{1}{-i\hbar}\left[\frac{-\hbar^2}{2m}\partial_x^2 \psi^* + V\psi^*\right]$$

V: has to be a real function $V^* = V$

$$(\partial_t \psi^*)\psi = \frac{-1}{i\hbar}\left[\frac{-\hbar^2}{2m}(\partial_x^2\psi^*)\psi + V\psi^*\psi\right]$$

$$\psi^*(\partial_t\psi) = \frac{1}{i\hbar}\left[\frac{-\hbar^2}{2m}\psi^*(\partial_x^2\psi) + \psi^* V\psi\right]$$

Because $V(x,t)$ just like $\psi(x,t)$

$$\psi^* V \psi = V\psi^*\psi$$

$$\psi^*(\partial_t\psi) = \frac{1}{i\hbar}\left[\frac{-\hbar^2}{2m}\psi^*(\partial_x^2\psi) + V\psi^*\psi\right]$$

$$(\partial_t \psi^*)\psi + \psi^*\partial_t\psi = \frac{-1}{i\hbar}\left[\frac{-\hbar^2}{2m}(\partial_x^2\psi^*)\psi + V\psi^*\psi + \frac{\hbar^2}{2m}\psi^*(\partial_x^2\psi) - V\psi^*\psi\right]$$

$$= -\frac{1}{i\hbar}\left[\frac{-\hbar^2}{2m}(\partial_x^2\psi^*)\psi + \frac{\hbar^2}{2m}\psi^*\partial_x^2\psi\right]$$

$$\frac{1}{i} = -i \Rightarrow \quad i^2 = -1$$

$$i\,i = -1$$

$$i = -1/i$$

$$\frac{\partial}{\partial t}|\psi|^2 = \frac{i\hbar}{2m}\left[\psi^*\partial_x^2\psi - (\partial_x^2\psi^*)\psi\right]$$

$$= \partial_x f(x)$$

we need "factorize" one ∂_x

$$\partial_x\left[\psi^*\partial_x\psi - (\partial_x\psi^*)\psi\right] = (\partial_x\psi^*)(\partial_x\psi) + \psi^*\partial_x^2\psi - (\partial_x^2\psi^*)\psi - (\partial_x\psi^*)(\partial_x\psi)$$

$$= \psi^*\partial_x^2\psi - (\partial_x^2\psi^*)\psi$$

$$\partial_t |\psi|^2 = \frac{i\hbar}{2m} \partial_x \left[\psi^* \partial_x \psi - (\partial_x \psi^*) \psi \right]$$

with this

$$\langle p \rangle = m \int x \frac{\partial}{\partial t} |\psi|^2 \, dx$$

$$= m \int x \frac{i\hbar}{2m} \partial_x \left[\psi^* \partial_x \psi - (\partial_x \psi^*) \psi \right] dx$$

$$\boxed{\langle p \rangle = \frac{i\hbar}{2} \int x \, \partial_x \left[\psi^* \partial_x \psi - (\partial_x \psi^*) \psi \right] dx}$$

Integration by parts

$u(x)$ and $v(x)$: two functions

$$\frac{\partial}{\partial x}(uv) = \left(\frac{\partial}{\partial x} u\right) v + u \frac{\partial}{\partial x} v \quad : \text{ chain rule}$$

$$(uv)' = u'v + uv'$$

$$u'v = -uv' + (uv)'$$

$$\int_a^b u'v = -\int_a^b uv' + \int_a^b (uv)'$$

$$= -\int_a^b uv' + uv \Big|_a^b$$

Identifying

$$v = x \qquad u = \left[\psi^* \partial_x \psi - (\partial_x \psi^*) \psi \right]$$

Assuming that $\psi(x \to \pm\infty) \to 0$

$$uv \Big|_{x=-\infty}^{\infty} = 0$$

$$\int u'v = -\int uv'$$

$$v' = \frac{\partial}{\partial x} v = 1$$

$$\langle p \rangle = -\frac{i\hbar}{2}\int\left[\psi^*\partial_x\psi - (\partial_x\psi^*)\psi\right]dx$$

we can reduce this expression further

- Integration by parts again

$$\int(\partial_x\psi^*)\psi\, dx = \int u'v \qquad v=\psi \quad u=\psi^*$$

$$= -\int uv' + uv\Big|$$

$$= -\int \psi^*\partial_x\psi$$

$$\langle p \rangle = -\frac{i\hbar}{2}\int\left[\psi^*\partial_x\psi + \psi^*\partial_x\psi\right]dx$$

$$= -i\hbar\int\psi^*\partial_x\psi \; dx$$

Remember : If we have an operator Ω

$$\langle\Omega\rangle = \int\psi^*\Omega\psi\, dx$$: we sandwich the operator Ω with the ψ^* and ψ

$$\langle p \rangle = \int\psi^*(-i\hbar\partial_x)\psi\, dx = \int\psi^* p\,\psi\, dx$$ the order is fundamental

Therefore the momentum operator is

$$\boxed{p = -i\hbar\,\partial_x}$$ Coordinate representation of the momentum operator

In 3D

$$p_x = -i\hbar\partial_x \qquad \vec{p} = -i\hbar\vec{\nabla}$$
$$p_y = -i\hbar\partial_y$$
$$p_z = -i\hbar\partial_z \qquad \vec{\nabla} = (\partial_x, \partial_y, \partial_z)$$

Having a dynamical variable that depends on position and momentum $\rightarrow Q(x,p)$

Q.M. mean value

$$\langle Q(x,p)\rangle = \int \psi^*(x,t)\, Q(x,-i\hbar\partial_x)\, \psi(x,t)\, dx$$

Schrödinger's equation

$$i\hbar\, \partial_t \psi = -\frac{\hbar^2}{2m}\partial_x^2 \psi + V\psi = \left(-\frac{\hbar^2}{2m}\partial_x^2 + V\right)\psi$$

$$p = -i\hbar\partial_x \Rightarrow p^2 = -\hbar^2\partial_x^2 \Rightarrow \frac{p^2}{2m} = -\frac{\hbar^2}{2m}\partial_x^2$$

In S.E.

$$i\hbar\,\partial_t\psi = \left(\frac{p^2}{2m} + V\right)\psi$$

$$= H\psi$$

$p = mv$

$p^2 = m^2v^2 \Rightarrow \frac{p^2}{2m} = \frac{1}{2}mv^2$

kinetic energy

H: Hamiltonian of the system

- Total energy of the system

$$i\hbar\,\partial_t\psi = H\psi$$

Not all operators are as trivial as x.

we will see that $px = \partial_x x$ while $xp = x\partial_x$