Operators in the Hilbert space

$\vec{v}$, $f(x)$, Matrices

Functions of vectors

$f(\alpha|v\rangle) = \alpha f(|v\rangle)$: function is linear

$\alpha \in \mathbb{C}$ $\quad |v\rangle \in V$

$\hat{T}$: operator

$$\hat{T} \sum_{i=1}^{n} \alpha_i |e_i\rangle = \sum_{i=1}^{n} \alpha_i \hat{T}|e_i\rangle \qquad \alpha_i \in \mathbb{C}$$

$$\hat{T}|e_i\rangle = \sum_{j=1}^{n} T_{ij}|e_j\rangle$$

$|e_i\rangle$: basis of the space V

$$\rightarrow \hat{T} \sum_{i=1}^{n} \alpha_i |e_i\rangle = \sum_{i=1}^{n} \alpha_i \hat{T}|e_i\rangle$$

$$= \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{n} T_{ij}|e_j\rangle$$

$$= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \alpha_i T_{ij} \right) |e_j\rangle$$

$$= \sum_{j=1}^{n} \beta_j |e_j\rangle$$

Transformed vector

$\beta_j = \sum_{i=1}^{n} \alpha_i T_{ij}$ Only depends on j

$\hat{T}$ has matrix form

$$\hat{T} = \begin{pmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & & \vdots \\ T_{n1} & T_{n2} & & T_{nn} \end{pmatrix}$$

finite dimension

Operator has infinite dimension (number of photons in a box)

$$\hat{T} = \begin{pmatrix} T_{11} & T_{12} & \cdots & \\ T_{21} & \ddots & & \\ \vdots & & \ddots & \\ \vdots & & & \ddots \end{pmatrix}$$ the matrix just goes on

Think $\hat{x}$ or $\hat{p}$

↳ both continuous $\qquad$ $\underline{\underline{|x\rangle}}$: basis vector

$\hat{x}|x\rangle = x|x\rangle \qquad \hat{p}|x\rangle = -i\hbar\partial_x|x\rangle$

we cannot write these operators as a matrix!

$|u\rangle = \sum_{i=1}^{n} \alpha_i |e_i\rangle \quad \Rightarrow \quad \underline{|v\rangle} = \hat{T}|u\rangle$

Particular vector $|f\rangle$ such that

$$\hat{T}|f\rangle = \underline{\underline{\lambda}}|f\rangle \qquad \lambda \in \mathbb{C}.$$

Example: $\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$|+\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \hat{\sigma}_x|+\rangle = +1|+\rangle \qquad \lambda_+ = +1$

$|-\rangle = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad \hat{\sigma}_x|-\rangle = -1|-\rangle \qquad \lambda_- = -1$

$|+\rangle, |-\rangle$ are the eigenvectors

λ_+, λ_- are the eigenvalues $\Rightarrow$ turn operators into scalars!

$\hat{T}|f\rangle = \lambda|f\rangle \Rightarrow (\hat{T} - \lambda\mathbb{1})|f\rangle = 0$

$\mathbb{1}$: Id. matrix with $\dim(\hat{T})$

$|f\rangle = 0$ is a trivial solution

↳ it works indep. $(\hat{T} - \lambda \mathbb{1})$

$|f\rangle = \cancel{\frac{1}{(\hat{T} - \lambda \mathbb{1})}} |0\rangle$ not possible

we need $\hat{T} - \lambda \mathbb{1}$ to be singular (cannot be inverted)

If $(\hat{T} - \lambda \mathbb{1})$ is singular $\Rightarrow \det(\hat{T} - \lambda \mathbb{1}) = 0$

- polynomial eq. for λ
- Number of solutions (roots) = $\dim(T)$

= Number of eigenvectors of $\hat{T}$

Special group of operators: <u>Observables</u>

$\hat{O}$ are operators for which all their λ are real-valued

- Can be the result of a measurement on WF.
- $\hat{T} = \hat{T}^{\dagger}$ "dagger" $\quad \hat{T}^{\dagger} = (T^*)^T$

$$\hat{T} = \begin{pmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & & T_{2n} \\ \vdots & \vdots & & \vdots \\ T_{n1} & T_{n2} & & T_{nn} \end{pmatrix} \Rightarrow \hat{T}^{\dagger} = \begin{pmatrix} T_{11}^* & T_{21}^* & \cdots & T_{n1}^* \\ T_{12}^* & T_{22}^* & \cdots & T_{n2}^* \\ \vdots & \vdots & & \vdots \\ T_{1n}^* & T_{2n}^* & & T_{nn}^* \end{pmatrix}$$

$\hat{T} = \hat{T}^{\dagger}$: Hermitian (Hermite).