Discrete Prob. Distributions

Bernouilli distribution

→ two possible outcomes with probabilities p and q.

X: random variable

$P(X=0) = q$ $\qquad$ $p+q=1 \Rightarrow q=1-p$

$P(X=1) = p$

$$\langle X \rangle = 0 \cdot P(X=0) + 1\, P(X=1)$$
$$= 0 \cdot q + 1 \cdot p = p$$
$$\langle X^2 \rangle = p$$
$$\mathrm{Var}(X) = \langle X^2 \rangle - \langle X \rangle^2$$
$$= p - p^2 = p(1-p) = pq$$

Binomial distribution

N independent Bernouilli random variables B_i

$$X = B_1 + B_2 + \cdots + B_N$$

Tossing of N biased coins; with X the number of Heads that we get

What is the distribution of X?

$$\begin{array}{ccccccc} H & H & T & H & T & T & T \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ p & p & q & p & q & q & q \end{array} = p^3 q^4 :$$

$X = 3$.

Probability of having 3 H out of 7 tosses other sequences also have this prob.

HHHTTTT $\rightarrow p^3 q^4$ pppqqqq

TTTTHHH $\rightarrow p^3 q^4$ qqqqppp

Concept of the Binomial

$$(a+b)^n = (a+b)(a+b) \cdots (a+b)$$

$\hookrightarrow$ n-3 times

$$= a^n + n\, a^{n-1} b + \cdots$$

The first element

$a^n = a \cdot a \cdot \cdots \cdot a$ $\Rightarrow$ there's only one sequence

aaaaaab $\Rightarrow$ 7 ways to accommodate b

$a^{n-1} b$ $\Rightarrow$ n ways to accommodate b

$a^{n-2} b^2$ $\Rightarrow$ $\frac{n(n-1)}{2}$ ways to accommodate two b's

To accommode k b's we have

$n(n-1) \cdots (n-k-1)$ options

$$n(n-1) \cdots (n-k-1) = \frac{n!}{(n-k)!}$$

$n!$: n factorial

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 1$$

There are k instances of b, we have k! ways to arrange them

In total, the number of distinguishable configurations is

$$\frac{n!}{(n-k)!}\frac{1}{k!} \equiv \binom{n}{k}$$

"Binomial coeffient of n and k"

Number of ways to choose k indistinguishable objects out of n.

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

Pascal's triangle

Back to our example

$$P(X=k) = \binom{N}{k} p^k q^{N-k} = \binom{N}{k} p^k (1-p)^{N-k}$$

Prob. having k Heads

Show that $\langle X \rangle = np$

$\mathrm{Var}(X) = np(1-p)$

The Bernouilli distribution is a particular case of the Binomial distribution when we perform only a single experiment.

Geometric distribution

How many times should we toss the coin until we have a Heads?

$TTTTTH \Rightarrow q^5 p = (1-p)^5 p$

$$P(X=k) = q^k p = (1-p)^k p$$

k: can be arbitrarily large ⇒ the dist. has infinite support

The distribution still needs to be normalized

$$\sum_{k=0}^{\infty} P(X=k) = 1$$ Check that the dist. is normalized

$$\langle X \rangle = \frac{p}{1-p} \qquad var(X) = \frac{p}{(1-p)^2}$$

Poisson distribution

Binomial distribution in the limit in which

$p \to 0$ but $n \to \infty$

$\lambda = pn$ is finite

λ: average number of success

$$\lim_{n\to\infty} P(X=k) = \lim_{n\to\infty} \binom{n}{k} p^k (1-p)^{n-k}$$ $p = \frac{\lambda}{n}$

$$= \lim_{n\to\infty} \frac{n!}{k!(n-k)!} \left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{n-k}$$

$$= \lim_{n\to\infty} \frac{1}{k!} \frac{n!}{(n-k)!} \lambda^k \left(\frac{1}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-k}$$

$$= \frac{\lambda^k}{k!} \lim_{n\to\infty} \frac{n!}{(n-k)!} \left(\frac{1}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-k}$$

If $\lim_{x\to a} f(x) = f(a)$ and $\lim_{x\to a} g(x) = g(a)$ then $\lim_{x\to a} f(x)g(x) = f(a)g(a)$

$$\lim_{n\to\infty} \frac{n!}{(n-k)!} \left(\frac{1}{n}\right)^k = \lim_{n\to\infty} \frac{n(n-1)(n-2)\cdots(n-k+1)}{n\cdot n\cdot n \cdots n}$$

$$= \lim_{n\to\infty} \frac{n}{n} \frac{n-1}{n} \frac{n-2}{n} \cdots \frac{(n-k+1)}{n} = 1$$

Euler identity $\quad e = \lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x$

with $n = \lambda/p$

$$\lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^n = e^{-\lambda}$$

$x = \frac{n}{\lambda} \quad \rightarrow \quad n = \lambda x \quad$ If $n \to \infty$, because λ is finite $\Rightarrow x \to \infty$

$$\lim_{x\to\infty} \left(1 - \frac{1}{x}\right)^{\lambda x}$$

Replace $x \to -y$

$$\lim_{y\to\infty} \left(1 + \frac{1}{y}\right)^{-\lambda y} = \left[\lim_{y\to\infty} \left(1 + \frac{1}{y}\right)^y\right]^{-\lambda}$$

$$= e^{-\lambda}$$

$$\lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^{-k} \approx 1^{-k} = 1$$

$$\boxed{P(\underline{X} = k) = e^{-\lambda} \frac{\lambda^k}{k!}}$$ Poisson distribution

$\langle \underline{X} \rangle = \lambda \ ; \ \mathrm{var}(\underline{X}) = \lambda$

Poisson burst $P(\underline{X} \geq 2)$