The Hilbert Space

FIELD set where $+, -, \times, \div$ are defined

$\mathbb{R}$: reals

$\mathbb{Q}$: rational p/q ; where $p, q \in \mathbb{Z}$

$\mathbb{C}$: complex

Other structures

↳ Set matrices!

A, B are matrices $\Rightarrow A + B$

$AB \Rightarrow$ Not element by element

B is the inverse of A

$$AB = BA = \mathbb{1}$$

where $\mathbb{1}$ is the identity matrix
$$\begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{pmatrix}$$

$$A \; \text{adj}(A) = \det(A) \; \mathbb{1} \quad \Rightarrow \quad A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$$

adj(A): Adjoint of A — Matrix

det(A): determinant of A — Scalar

A^{-1} exist iff $\det(A) \neq 0$

↳ if and only if

If $\det(A) = 0 \Rightarrow A$ is a SINGULAR MATRIX

VECTOR SPACE

- It consist of a set V of objects called vectors, along with another set F of numbers (a field) that are called scalars
- The vectors in V can be added

$v_1, v_2 \in V \Rightarrow v_1 + v_2 \in V$

- The vectors in V can be scaled by elements of F

$v_1 \in V$ and $\alpha \in F \Rightarrow \alpha v_1 \in V$

Example: Geometrical vectors

$\vec{v} = (v_1, v_2, \dots, v_N)$: geometry in a N-dimensional vector space

tuple of numbers

In 3D $\vec{r} = (x, y, z)$

N can be infinite ⇒ Let's keep it general!

Funtion Spaces

Polynomials of order up to N: they form a vector space P_N

P_2 $\quad f_1(x) = x^2 \qquad f_2(x) = -3x + 1$

$f_1, f_2 \in P_2 \qquad f_1(x) + f_2(x) = x^2 - 3x + 1 \in P_2$

$\alpha f_1 + \beta f_2 = \alpha x^2 - 3\beta x + \beta \in P_2$

$f_1(x) f_2(x) = -3x^3 + x^2 \notin P_2$ These are not a field

In Q.M. we write $|u\rangle$ ket

$|i\rangle = x^i$ for the P_N

simply notation

Basis vectors: Collection of vectors in V such tha every element of V can be expressed as a linear combination of them.

If $|e_i\rangle$ form a basis: $|u\rangle = \sum_{i=1}^{n} \alpha_i |e_i\rangle$

for some scalars α_i

coordinates of $|u\rangle$ in the basis $|e_i\rangle$

$\hat{\imath}, \hat{\jmath}, \hat{k}$: $\vec{r} = x_0 \hat{\imath} + y_0 \hat{\jmath} + z_0 \hat{k}$

Vectors are linearly independent when they cannot be written as a combination of the other.

$|i\rangle = x^i$ are linearly ind. because x^i cannot be written as combination of polynomials x^j with $j \neq i$

Number of lin. ind. vector = dimension of the space

- 3D space ⇒ 3 line. ind. vectors
- A space can have ∞ dim. ⇒ P_∞

Conjugate vector

$|u\rangle^{\dagger}$ "dagger"

$\underset{\text{ket}}{|u\rangle}^{\dagger} \equiv \langle u|$ bra

$\langle u|v\rangle \in F$: a scalar

bra-ket

Q.M. is defined over a complex field $\langle u|v\rangle \in \mathbb{C}$

$\langle u|v\rangle^{*} = \langle v|u\rangle$

$\langle u|u\rangle \geq 0$ it is $=0$ iff $|u\rangle = |0\rangle$

$|0\rangle \neq 0$

In 3D : $|0\rangle$ origin of coordinates

Q.M: $|0\rangle$ is the Vacuum

For geometric vectors

$\langle u|v\rangle = \vec{u}\cdot\vec{v} = \sum_i^N u_i v_i$

For function space

$\langle f|g\rangle = \int f^{*}(x)\, g(x)\, dx$ overlap integral

$\mathbb{C}$ also has the structure of a complex vector space

$$|u\rangle = \sum_{i=1}^{n} \alpha_i |e_i\rangle \qquad \langle u| = \sum_{j=1}^{n} \alpha_j^* \langle e_j|$$

$$\langle u|u\rangle = \sum_{j=1}^{n} \alpha_j^* \langle e_j| \sum_{i=1}^{n} \alpha_i |e_i\rangle$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{n} \alpha_j^* \alpha_i \langle e_j|e_i\rangle$$

orthonormal

$|e_i\rangle$ are linearly indep. and orthogonal

$\langle e_i|e_j\rangle = 0$ unless $i=j$

$\langle e_i|e_i\rangle = 1 \rightarrow |e_i\rangle$ are normalized

$$\langle e_i|e_j\rangle = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i=j \end{cases}$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{n} \alpha_i \alpha_j^* \delta_{ij}$$

$\vec{r} = (x, y, z)$

$\vec{r}\cdot\vec{r} = x^2+y^2+z^2 \in \mathbb{R}$

$\sqrt{\vec{r}\cdot\vec{r}} = |\vec{r}|$

$$\langle u|u\rangle = \sum_{i=1}^{n} \alpha_i \alpha_i^* = \sum_{i=1}^{n} |\alpha_i|^2$$

$\| |u\rangle \| = \sqrt{\langle u|u\rangle}$: Norm of $|u\rangle$

For function space

$|i\rangle = x^i$ ~~orthogonal normal~~

Legendre polynomials are orthogonal $-1 < x < -1$

$P_n(x)$

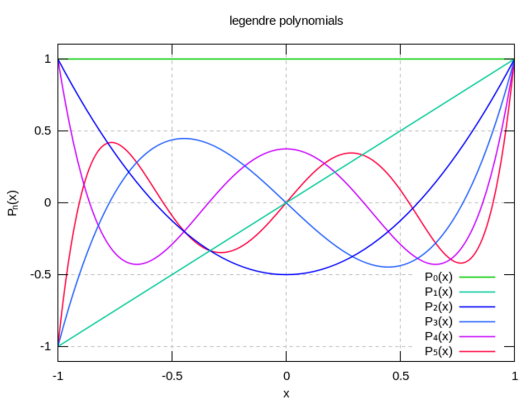
$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = (3x^2-1)/2$

$P_3(x) = (5x^2 - 3x)/2$

$P_4(x) = (35x^4 - 30x^2 + 3)/8$



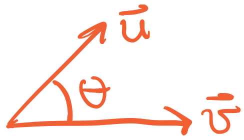
Check that

$$\int_{-1}^{1} P_n(x) P_m(x)\, dx = 0 \quad \text{if } m \neq n$$

for a few n and m

$\vec{u}\cdot\vec{v} = |\vec{u}|\,|\vec{v}|\cos\theta \le |\vec{u}|\,|\vec{v}|$

The interpretation is not clear in general, but it remains true



$$\langle u|u\rangle\langle v|v\rangle \ge |\langle u|v\rangle|^2$$

Cauchy-Schwarz inequality

Such a vector space becomes a Hilbert space when it also a Banach space.

- Complete: limiting process $\lim_{x\to 0} e^{-x} = 1$
  Limit is also a vector in the space

There are no "holes" or missing vectors
↳ we can do calculus and it will work

Q.M. space is a Hilbert space!