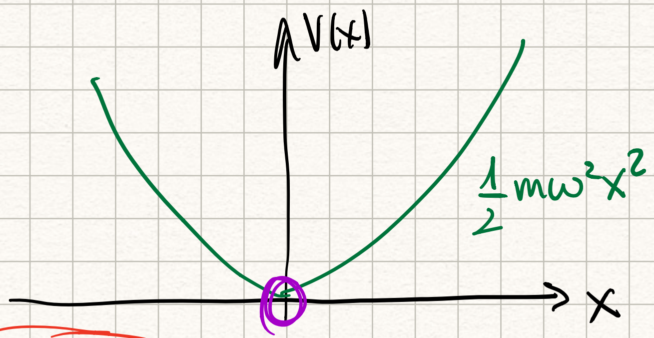
Solutions of the H.O.

$$H = \left(a_+ a_- + \frac{1}{2}\hbar\omega\right) \qquad H\psi = E\psi$$

$V(x) =$ whatever function $\qquad E > \min(V(x))$

solution is physical.



$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 \quad \text{classical mech.}$$

$$E > 0$$

$$H\psi = E\psi \Rightarrow \phi = a_-\psi \Rightarrow H\phi = (E - \hbar\omega)\phi$$

$$H(a_-^k\psi) = \underbrace{(E - k\hbar\omega)}_{E_k > 0}(a_-^k\psi)$$

finite

$$E - k\hbar\omega \geq 0$$

$$\frac{E}{\hbar\omega} \geq k$$

$$k = \underbrace{\frac{E}{\hbar\omega} + \varepsilon}_{\text{integer}} \qquad 0 < \varepsilon < 1$$

$$\Rightarrow E - k\hbar\omega = E - \left(\frac{E}{\hbar\omega} + \varepsilon\right)\hbar\omega$$
$$= E - E - \varepsilon\hbar\omega$$
$$= -\varepsilon\hbar\omega < 0$$

There's a maximum k that we can apply the annihilation operator.

$a_-\psi_0 = 0$: scalar zero $\qquad \psi_0$: lowest energy eigenvector of H.

$$\psi_1 = a_+\psi_0$$
$$\psi_2 = a_+\psi_1 = a_+^2\psi_0$$
$$\vdots$$
$$\psi_n = a_+\psi_{n-1} = a_+^n\psi_0$$

$$a_\pm = \frac{-i}{\sqrt{2m}}\left(-\hbar\partial_x \pm m\omega x\right)$$

$$\boxed{a_- \psi_0 = 0} \qquad 0 = \frac{-i}{\sqrt{2m}}\left(-\hbar\partial_x - m\omega x\right)\psi_0$$

$$0 = -\hbar\partial_x\psi_0 - m\omega x\psi_0$$

$$-\hbar\partial_x\psi_0 = m\omega x\psi_0$$

$$-\frac{1}{\psi_0}\,\hbar\frac{\partial\psi_0}{\partial x}dx = m\omega x\psi_0\,\frac{1}{\psi_0}\,dx$$

$$-\hbar\int\frac{\partial\psi_0}{\psi_0} = m\omega\int x\,dx$$

$$\int\frac{dx}{x} = \ln x \qquad \int x\,dx = \frac{1}{2}x^2 + C$$

$$-\hbar\ln\psi_0 = \frac{1}{2}m\omega x^2 + C$$

$$\ln\psi_0 = -\frac{m\omega}{2\hbar}x^2 - \frac{C}{\hbar}$$

$$e^{a+b} = e^a e^b$$

$$\psi_0 = \exp\left[-\frac{m\omega}{2\hbar}x^2 - C\right]$$

$$= \exp\left[\frac{-m\omega}{2\hbar}x^2\right]e^{-C}$$

$$\boxed{\psi_0 = A\exp\left[-\frac{m\omega}{2\hbar}x^2\right]}$$

$$a_-\psi_0 = 0$$

$$H\psi_0 = \left(a_+a_- + \frac{1}{2}\hbar\omega\right)\psi_0 = a_+\cancel{a_-\psi_0}^{\,0} + \frac{1}{2}\hbar\omega\psi_0$$

$$= \frac{1}{2}\hbar\omega\psi_0 = E_0\psi_0$$

$$\boxed{E_0 = \frac{1}{2}\hbar\omega}$$

$$\Psi(x,t) = \psi_0(x)\,e^{-iE_0t/\hbar}$$

$$= \psi_0(x)\,e^{-i\omega t/2}$$

$$\frac{E_0}{\hbar} = \frac{1}{2}\omega$$

$$\boxed{\int_{-\infty}^{\infty}e^{-\alpha x^2}dx = \sqrt{\frac{\pi}{\alpha}}}$$

→ Gaussian function

$\psi_0(\pm\infty) = 0$

Obtain the value of A

$$1 = \int_{-\infty}^{\infty} |\psi_0(x)|^2 dx = \int_{-\infty}^{\infty} A^2 \exp\left[-\frac{m\omega}{\hbar}x^2\right] = A^2\sqrt{\frac{\pi}{m\omega/\hbar}} = A^2\sqrt{\frac{\hbar\pi}{m\omega}}$$

$$A^2 = \sqrt{\frac{m\omega}{\pi\hbar}} \Rightarrow A = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$$

$$\boxed{\psi_0(x,t) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left[-\frac{m\omega x^2}{2\hbar}\right] e^{-i\omega t/2}}$$

Find the 1st excited state $\Rightarrow$ nth excited state

$$a_+ \psi_0(x) = \frac{-i}{\sqrt{2m}}\left(-\hbar\partial_x + m\omega x\right)\psi_0$$

$$= \frac{i}{\sqrt{2m}}\left(\hbar\partial_x - m\omega x\right)\psi_0$$

$$\left(\hbar\partial_x - m\omega x\right)\psi_0$$

$$\partial_x\psi_0 = \partial_x\left(A e^{-\frac{m\omega x^2}{2\hbar}}\right)$$

$$= -\frac{m\omega}{\hbar} x A e^{-\frac{m\omega x^2}{2\hbar}}$$

$$\hbar\partial_x\psi_0 = -m\omega x\psi_0$$

$$\hbar\partial_x\psi_0 - m\omega x\psi_0 = -m\omega x\psi_0 - m\omega x\psi_0$$

$$= -2m\omega x\psi_0$$

$$= -2m\omega x A e^{-\frac{m\omega x^2}{2\hbar}}$$

$$\psi_1(x) = A_1 x \exp\left[-\frac{m\omega x^2}{2\hbar}\right]$$

$$\psi_1 = a_+\psi_0$$

$$H\psi_0 = \frac{1}{2}\hbar\omega\psi_0$$

$$H(a_+\psi_0) = \left(\frac{1}{2}\hbar\omega + \hbar\omega\right)a_+\psi_0$$

$$H\psi_1 = \frac{3}{2}\hbar\omega\psi_1 \qquad E_1 = \frac{3}{2}\hbar\omega$$

$$\psi_1(x,t) = A_1 x \exp\left[-\frac{m\omega x^2}{2\hbar}\right] e^{-i3\omega t/2}$$

find by normalizing the W.F.

$\psi_2 \propto a_+ \psi_1$

$(\hbar\partial_x - m\omega x)\psi_1$

$$\partial_x \psi_1 = \partial_x \left[x e^{-m\omega x^2/2\hbar}\right]$$
$$= e^{-m\omega x^2/2\hbar} - \frac{m\omega x^2}{\hbar} e^{-m\omega x^2/2\hbar}$$
$$= \left[1 - \frac{m\omega x^2}{\hbar}\right] e^{-m\omega x^2/2\hbar}$$

$$m\omega x \psi_1 = m\omega x^2 e^{-m\omega x^2/2\hbar}$$

$$(\hbar\partial_x - m\omega x)\psi_1 = \hbar\left[1 - \frac{m\omega x^2}{\hbar}\right] e^{-m\omega x^2/2\hbar} - m\omega x^2 e^{-m\omega x^2/2\hbar}$$
$$= \left[\hbar - m\omega x^2 - m\omega x^2\right] e^{-m\omega x^2/2\hbar}$$
$$= \hbar\left[1 - \frac{2m\omega x^2}{\hbar}\right] e^{-m\omega x^2/2\hbar}$$

$$\psi_2(x) = A_2 \left[1 - \frac{2m\omega x^2}{\hbar}\right] e^{-m\omega x^2/2\hbar}$$

find by normalizing the W.F.

$$E_2 = \frac{5}{2}\hbar\omega = \left(\frac{3}{2}\hbar\omega + \hbar\omega\right) = \left(E_1 + \underline{\hbar\omega}\right)$$

$$\Psi_2(x,t) = A_2 \left[1 - \frac{2m\omega x^2}{\hbar}\right] e^{-m\omega x^2/2\hbar} \; e^{-i5\omega t/2}$$

Dimensionless variable $\xi = \sqrt{\frac{m\omega}{\hbar}}\, x$ $\qquad [\xi] = 1$

$[x] = m$

$$\psi_0(\xi) = a_0 e^{-\xi^2/2}$$
$$\psi_1(\xi) = \psi_0'(\xi) = \frac{\partial}{\partial \xi}\left[a_0 e^{-\xi^2/2}\right] = a_0 \frac{\partial}{\partial \xi}\left[e^{-\xi^2/2}\right]$$
$$= a_0(-\xi) e^{-\xi^2/2}$$
$$= -\xi a_0 e^{-\xi^2/2} = -\xi \psi_0$$

$$\psi_2(\xi) = \psi_1'(\xi) = \psi_0''(\xi)$$

$$= \frac{\partial}{\partial \xi}(-\xi\psi_0) = -\psi_0 - \xi \frac{\partial}{\partial \xi}\psi_0 = \psi_1(\xi) = -\psi_0 - \xi\psi_1$$

$$= -\psi_0 - \xi(-\xi\psi_0) = (-1 + \xi^2)\psi_0$$

$$= -(1-\xi^2)\psi_0$$

$$\psi_3(\xi) = \psi_2'(\xi) = -\frac{\partial}{\partial \xi}\left[(1-\xi^2)\psi_0\right] = -\left\{(-2\xi)\psi_0 + (1-\xi^2)\,\partial_\xi\psi_0\right\}$$

$$= -\left\{-2\xi\psi_0 + (1-\xi^2)(-\xi)\psi_0\right\} = -\left\{-2\xi\psi_0 - \xi\psi_0 + \xi^3\psi_0\right\}$$

$$= -\left\{\xi^3 - 3\xi\right\}\psi_0$$

$$\psi_4(\xi) = (\xi^4 - 6\xi^2 + 3)\psi_0$$

$$\psi_5(\xi) = -(\xi^5 - 10\xi^3 + 15\xi)\psi_0$$

$$\psi_n(\xi) = -\underbrace{\text{Polynomial}\,(\xi^n)}_{\text{Hermite Polynomials}}\,\psi_0$$

$H_0(x) = 1$

$H_1(x) = 2x$

$H_2(x) = 4x^2 - 2$

$H_3(x) = 8x^3 - 12x$

$H_4(x) = 16x^4 - 48x^2 + 12$

$H_5(x) = 32x^5 - 160x^3 + 120x$

$$\Psi_n(x,t) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} \boxed{H_n\left(\sqrt{\frac{m\omega}{\hbar}}\,x\right)} \exp\left[-\frac{m\omega x^2}{2\hbar}\right] e^{-i(2n+1)\frac{\omega t}{2}}$$

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega \Rightarrow \frac{E_n}{\hbar} = (2n+1)\frac{\omega}{2}$$

$H_0\left(\sqrt{\frac{m\omega}{\hbar}}\,x\right) = 1$ $\quad$ $H_1\left(\sqrt{\frac{m\omega}{\hbar}}\,x\right) = 2\sqrt{\frac{m\omega}{\hbar}}\,x$

Lecture 10

$\psi_n \quad |\psi_n|^2$

Quantum tunnelling