Decay and incoherent excitation

$$\partial_t \rho = \frac{i}{\hbar}[\rho, H] + \frac{1}{2}\sum_k' \mathcal{L}_{L_k}\rho$$

$$\mathcal{L}_{L_k}\rho = \left(2L_k \rho L_k^\dagger - L_k^\dagger L_k \rho - \rho L_k^\dagger L_k\right)$$

The master eq. is for $\rho \rightarrow$ which generalizes $|\psi\rangle$
we can also find an equation for $\langle\hat{o}\rangle$

$$\langle \hat{o}\rangle(t) = \mathrm{Tr}_S\{\hat{o}\,\rho(t)\}$$
$$= \mathrm{Tr}\{\hat{o}\,\rho(t)\}$$

$$\hat{o}\,\partial_t \rho(t) = \partial_t(\hat{o}\,\rho(t))$$ $\hat{o}$: is time independent

$$\mathrm{Tr}\{\hat{o}\,\partial_t\rho(t)\} = \mathrm{Tr}\{\partial_t(\hat{o}\,\rho(t))\}$$
$$= \partial_t\left(\mathrm{Tr}\{\hat{o}\,\rho(t)\}\right)$$
$$= \partial_t \langle\hat{o}\rangle(t)$$

On the right-hand side we have

$$\hat{o}\left(\frac{i}{\hbar}[\rho, H] + \frac{1}{2}\sum_k' \mathcal{L}_{L_k}\rho\right)$$

By steps

1. $$\frac{i}{\hbar}\hat{o}[\rho, H] = \frac{i}{\hbar}\hat{o}(\rho H - H\rho) = \frac{i}{\hbar}(\hat{o}\rho H - \hat{o}H\rho)$$

$$\mathrm{Tr}\left\{\frac{i}{\hbar}\hat{o}[\rho, H]\right\} = \frac{i}{\hbar}\left(\mathrm{Tr}\{\hat{o}\rho H\} - \mathrm{Tr}\{\hat{o}H\rho\}\right)$$

Property of the trace

$$\mathrm{Tr}\{A_1 A_2 \cdots A_n\} = \mathrm{Tr}\{A_n A_1 A_2 \cdots A_{n-1}\}$$
$$= \mathrm{Tr}\{A_2 \cdots A_n A_1\}$$

$$\mathrm{Tr}\{\hat{O}\rho H\} = \mathrm{Tr}\{H\hat{O}\rho\}$$

$$\mathrm{Tr}\left\{\frac{i}{\hbar}\hat{O}[\rho,H]\right\} = \frac{i}{\hbar}\left(\mathrm{Tr}\{H\hat{O}\rho\} - \mathrm{Tr}\{\hat{O}H\rho\}\right)$$
$$= \frac{i}{\hbar}\left(\langle H\hat{O}\rangle - \langle \hat{O}H\rangle\right)$$
$$= \frac{i}{\hbar}\mathrm{Tr}\{(H\hat{O} - \hat{O}H)\rho\}$$
$$= \frac{i}{\hbar}\langle H\hat{O} - \hat{O}H\rangle = \frac{i}{\hbar}\langle [H,\hat{O}]\rangle$$

$$\partial_t \langle \hat{O}\rangle(t) = \frac{i}{\hbar}\langle [H,\hat{O}]\rangle$$ in agreement with Heisenberg eq.!

② $$\hat{O}\,\frac{1}{2}\sum_k \mathcal{L}_{L_k}\rho = \frac{1}{2}\sum_k \hat{O}\left(2L_k\rho L_k^\dagger - L_k^\dagger L_k\rho - \rho L_k^\dagger L_k\right)$$

$$\mathrm{Tr}\left\{\hat{O}\,\frac{1}{2}\sum_k \mathcal{L}_{L_k}\rho\right\} =$$
$$= \frac{1}{2}\sum_k\left(2\,\mathrm{Tr}\{\hat{O}L_k\rho L_k^\dagger\} - \mathrm{Tr}\{\hat{O}L_k^\dagger L_k\rho\} - \mathrm{Tr}\{\hat{O}\rho L_k^\dagger L_k\}\right)$$
$$= \frac{1}{2}\sum_k\left(2\,\mathrm{Tr}\{L_k^\dagger\hat{O}L_k\rho\} - \mathrm{Tr}\{\hat{O}L_k^\dagger L_k\rho\} - \mathrm{Tr}\{L_k^\dagger L_k\hat{O}\rho\}\right)$$
$$= \frac{1}{2}\sum_k\left(2\langle L_k^\dagger\hat{O}L_k\rangle - \langle\hat{O}L_k^\dagger L_k\rangle - \langle L_k^\dagger L_k\hat{O}\rangle\right)$$

Putting everything together

$$\partial_t \langle o \rangle (t) = \frac{i}{\hbar} \langle [H, \hat{o}] \rangle +$$
$$+ \frac{1}{2} \sum_k \left(2 \langle L_k^\dagger \hat{o} L_k \rangle - \langle \hat{o} L_k^\dagger L_k \rangle - \langle L_k^\dagger L_k \hat{o} \rangle \right)$$

$\boxed{\text{Decay}}$ $\quad L_k = \sqrt{\gamma_\sigma}\, \sigma \qquad \sigma = |g\rangle\langle e|$

The $\rho(t)$ of a 2LS is given

$$\rho(t) = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{cc} |g\rangle & |e\rangle \\ \left(\begin{array}{c} 1 - \langle \sigma^\dagger \sigma \rangle \\ \\ \langle \sigma^\dagger \rangle \end{array}\right. & \left.\begin{array}{c} \langle \sigma \rangle \\ \\ \langle \sigma^\dagger \sigma \rangle \end{array}\right) \end{array} \begin{array}{c} \\ \langle g| \\ \\ \langle e| \end{array} \qquad \langle \sigma^\dagger \rangle = \langle \sigma \rangle^*$$

$$\langle \sigma \rangle^* = \left(\mathrm{Tr}\{\sigma \rho\} \right)^* = \mathrm{Tr}\{(\sigma\rho)^\dagger\} = \mathrm{Tr}\{\rho^\dagger \sigma^\dagger\}$$
$$= \mathrm{Tr}\{\rho \sigma^\dagger\} = \mathrm{Tr}\{\sigma^\dagger \rho\} = \langle \sigma^\dagger \rangle$$

Let's assume $\quad H = \hbar\omega_\sigma \sigma^\dagger \sigma + \hbar \Omega (\sigma + \sigma^\dagger)$

$$\partial_t \langle o \rangle = \frac{i}{\hbar} \langle [H, \hat{o}] \rangle + \frac{\gamma_\sigma}{2} \left(\langle \sigma^\dagger \hat{o} \sigma \rangle - \langle \sigma^\dagger \sigma \hat{o} \rangle - \langle \hat{o} \sigma^\dagger \sigma \rangle \right)$$

$\Rightarrow$ with $\hat{o} = \sigma^\dagger \sigma$

$$\boxed{\begin{array}{l} \sigma\sigma^\dagger + \sigma^\dagger\sigma = 1 \\ \sigma\sigma^\dagger = (1 - \sigma^\dagger\sigma) \end{array}}$$

$$\begin{aligned} H\hat{o} &= \hbar\omega_\sigma \sigma^\dagger\sigma\sigma^\dagger\sigma + \hbar\Omega(\sigma + \sigma^\dagger)\sigma^\dagger\sigma \\ &= \hbar\omega_\sigma \sigma^\dagger(1 - \sigma^\dagger\sigma)\sigma + \hbar\Omega\, \sigma\sigma^\dagger\sigma \\ &= \hbar\omega_\sigma \sigma^\dagger\sigma + \hbar\Omega(1 - \sigma^\dagger\sigma)\sigma \\ &= \hbar\omega_\sigma \sigma^\dagger\sigma + \hbar\Omega\sigma \end{aligned}$$

$$\hat{o} H = \hbar\omega_\sigma \sigma^\dagger\sigma + \hbar\Omega\sigma^\dagger$$ Check!

$[H, \hat{o}] = \hbar\Omega(\sigma - \sigma^+)$

- $\sigma^+ \sigma^+\sigma \sigma = 0$
- $\sigma^+\sigma \sigma^+\sigma = \sigma^+(1 - \sigma^+\sigma)\sigma = \sigma^+\sigma$
-

$$\partial_t \langle \sigma^+\sigma \rangle = \frac{i}{\hbar} \hbar\Omega \left(\langle \sigma \rangle - \langle \sigma^+ \rangle \right) - \gamma_\sigma \langle \sigma^+\sigma \rangle$$

$$\partial_t \langle \sigma^+\sigma \rangle = i\Omega \left(\langle \sigma \rangle - \langle \sigma^+ \rangle \right) - \gamma_\sigma \langle \sigma^+\sigma \rangle$$

$\rightarrow$ For $\hat{o} = \sigma$

$$\begin{aligned} H\hat{o} &= \hbar\omega_\sigma \sigma^+\sigma\sigma + \hbar\Omega(\sigma + \sigma^+)\sigma \\ &= \hbar\Omega\, \sigma^+\sigma \end{aligned}$$

$$\begin{aligned} \hat{o}H &= \hbar\omega_\sigma \sigma\sigma^+\sigma + \hbar\Omega\, \sigma(\sigma + \overset{\sigma\sigma^+}{\sigma^+}) \\ &= \hbar\omega_\sigma (1 - \sigma^+\sigma)\sigma + \hbar\Omega (1 - \sigma^+\sigma) \\ &= \hbar\omega_\sigma \sigma + \hbar\Omega - \hbar\Omega\, \sigma^+\sigma \end{aligned}$$

$$[H, \hat{o}] = 2\hbar\Omega\, \sigma^+\sigma - \hbar\omega_\sigma \sigma - \hbar\Omega$$

- $\sigma^+ \sigma \sigma = 0$
- $\sigma^+ \sigma \sigma = 0$
- $\sigma \sigma^+ \sigma = (1 - \sigma^+\sigma)\sigma = \sigma$

$$\boxed{\partial_t \langle \sigma \rangle = 2i\Omega \langle \sigma^+\sigma \rangle - i\omega_\sigma \langle \sigma \rangle - i\Omega - \frac{\gamma_\sigma}{2} \langle \sigma \rangle}$$

$$\partial_t \langle \sigma^+ \rangle = -2i\Omega \langle \sigma^+\sigma \rangle + i\omega_\sigma \langle \sigma^+ \rangle + i\Omega - \frac{\gamma_\sigma}{2} \langle \sigma^+ \rangle$$

Three coupled diff. eqs. for $\langle \sigma^+\sigma \rangle$, $\langle \sigma \rangle$ and $\langle \sigma^+ \rangle$.

Homework.
Find the solution when $\langle \sigma^+\sigma \rangle(0) = 1$
$\langle \sigma \rangle(0) = 0$

- Spontaneous emission $\Omega = 0$: no coherent driving

$$\partial_t \langle \sigma^\dagger \sigma \rangle = i\Omega\left(\langle \sigma \rangle - \langle \sigma^\dagger \rangle\right) - \gamma_\sigma \langle \sigma^\dagger \sigma \rangle$$

$$\partial_t \langle \sigma \rangle = 2i\Omega \langle \sigma^\dagger \sigma \rangle - i\omega_\sigma \langle \sigma \rangle - i\Omega - \frac{\gamma_\sigma}{2} \langle \sigma \rangle$$

$$\partial_t \langle \sigma^\dagger \rangle = -2i\Omega \langle \sigma^\dagger \sigma \rangle + i\omega_\sigma \langle \sigma^\dagger \rangle + i\Omega - \frac{\gamma_\sigma}{2} \langle \sigma^\dagger \rangle$$

If $\Omega = 0$

$$\partial_t \langle \sigma^\dagger \sigma \rangle = -\gamma_\sigma \langle \sigma^\dagger \sigma \rangle$$

$$\partial_t \langle \sigma \rangle = -\left(i\omega_\sigma + \frac{\gamma_\sigma}{2}\right) \langle \sigma \rangle$$

$$\partial_t \langle \sigma^\dagger \rangle = \left(i\omega_\sigma - \frac{\gamma_\sigma}{2}\right) \langle \sigma^\dagger \rangle$$

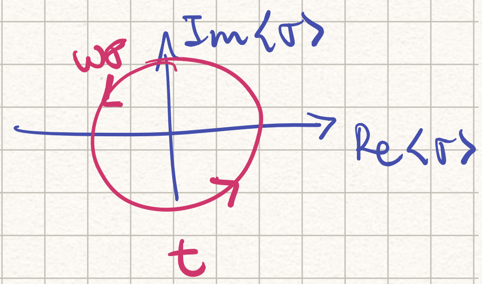
$$\partial_t x = -\alpha x$$

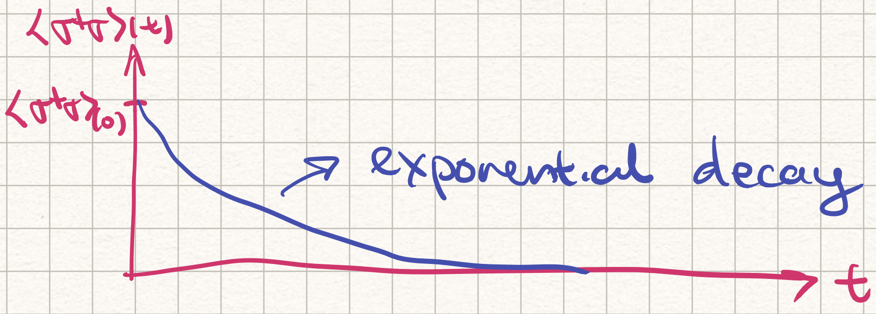
$$x(t) = x(0)\, e^{-\alpha t}$$

$$\langle \sigma^\dagger \sigma \rangle(t) = \langle \sigma^\dagger \sigma \rangle(0)\, e^{-\gamma_\sigma t}$$

$$\langle \sigma \rangle(t) = \langle \sigma \rangle(0)\, e^{-(i\omega_\sigma + \gamma_\sigma/2)t} = \langle \sigma \rangle(0)\, e^{-i\omega_\sigma t} e^{-\gamma_\sigma t/2}$$

$$\langle \sigma^\dagger \rangle(t) = \langle \sigma^\dagger \rangle(0)\, e^{i\omega_\sigma t} e^{-\gamma_\sigma t/2}$$





γ_σ : decay rate of the 2LS.

- Incoherent excitation $L_k = \sqrt{P_\sigma}\, \sigma^\dagger$

$$\frac{1}{2} \sum_k \left(2 \langle L_k^\dagger \hat{O} L_k \rangle - \langle \hat{O} L_k^\dagger L_k \rangle - \langle L_k^\dagger L_k \hat{O} \rangle \right)$$

For $\hat{O} = \sigma^\dagger \sigma$

- $\sigma \sigma^\dagger \sigma \sigma^+ = (1 - \sigma^\dagger \sigma)(1 - \sigma^\dagger \sigma)$

$$= 1 - \sigma^\dagger\sigma - \sigma^\dagger\sigma + \sigma^\dagger\sigma\sigma^\dagger\sigma$$
$$= 1 - 2\sigma^+\sigma + \sigma^\dagger(1 - \sigma^\dagger\sigma)\sigma$$
$$= 1 - 2\sigma^\dagger\sigma + \sigma^\dagger\sigma$$
$$= 1 - \sigma^\dagger\sigma$$

- $\sigma^\dagger \sigma \sigma \sigma^+ = 0$
- $\sigma \sigma^+ \sigma^\dagger \sigma = 0$

$$\partial_t \langle \sigma^\dagger \sigma \rangle = i\Omega(\langle\sigma\rangle - \langle\sigma^+\rangle) - \gamma_\sigma \langle \sigma^\dagger\sigma\rangle + P_\sigma(1 - \langle\sigma^\dagger\sigma\rangle)$$

For σ:

- $\sigma \sigma \sigma^+ = 0$
- $\sigma \sigma \sigma^+ = 0$
- $\sigma \sigma^\dagger \sigma = (1 - \sigma^\dagger\sigma)\sigma = \sigma$

$$\partial_t \langle\sigma\rangle = 2i\Omega\langle\sigma^\dagger\sigma\rangle - i\omega_\sigma\langle\sigma\rangle - i\Omega - \frac{\gamma_\sigma}{2}\langle\sigma\rangle - \frac{P_\sigma}{2}\langle\sigma\rangle$$

$$\partial_t \langle\sigma^+\rangle = -2i\Omega\langle\sigma^\dagger\sigma\rangle + i\omega_\sigma\langle\sigma^+\rangle + i\Omega - \frac{\gamma_\sigma}{2}\langle\sigma^+\rangle - \frac{P_\sigma}{2}\langle\sigma^+\rangle$$

Let's consider the case $\Omega = 0$

$$\partial_t \langle\sigma^\dagger\sigma\rangle = P_\sigma - (\gamma_\sigma + P_\sigma)\langle\sigma^\dagger\sigma\rangle$$

$$\partial_t \langle\sigma\rangle = \left(-i\omega_\sigma - \frac{\gamma_\sigma + P_\sigma}{2}\right)\langle\sigma\rangle$$

$$\partial_t \langle\sigma^+\rangle = \left(i\omega_\sigma - \frac{\gamma_\sigma + P_\sigma}{2}\right)\langle\sigma^+\rangle$$

$\boxed{\text{Homework: Find } \langle\sigma^+\sigma\rangle(t)}$

$$\langle\sigma\rangle(t) = \langle\sigma\rangle(0)\, e^{-i\omega_\sigma t}\, e^{-(\gamma_\sigma + P_\sigma)t/2}$$

$$\langle\sigma^+(t)\rangle = \langle\sigma^+(0)\rangle\, e^{i\omega_\sigma t}\, e^{-(\gamma_\sigma + P_\sigma)t/2}$$