Fermions: the 2-level system and Pauli matrices.

<u>Fermi:</u>

→ electrons
→ atoms
→ molecules

• spin ↑ , spin ↓ : 2 states

$\{|\uparrow\rangle, |\downarrow\rangle\}$; $\{|+\rangle, |-\rangle\}$

Hilbert space has dim = 2.

$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $|-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$|\psi\rangle = \alpha|+\rangle + \beta|-\rangle$, $|\alpha|^2 + |\beta|^2 = 1$, $\alpha, \beta \in \mathbb{C}$

$= \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

→ most general W.F.

Any observable can be expressed as alinear comb.

Pauli Matrices: σ_i

$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

• They are Hermitian: $\sigma_i^+ = \sigma_i$; $i = x, y, z$

• $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \mathbb{1} = -i\,\sigma_x \sigma_y \sigma_z$ Homework

• $\mathrm{Tr}(\sigma_i) = 0$: Traceless

• $\det(\sigma_i) = -1$

"Quantum algebra"

$\boxed{[a, a^+] = 1 \text{ : Commutator} \quad \text{Bosons}}$

Pauli Matrices:

- $[\sigma_a, \sigma_b] = 2i\, \sigma_c\, \varepsilon_{abc}$

ε_{abc} : Levi-Civita symbol

$$\varepsilon_{abc} = \begin{cases} 1 & (a,b,c) = \{(x,y,z),(y,z,x),(z,x,y)\} \\ -1 & (a,b,c) = \{(z,y,x),(y,x,z),(x,z,y)\} \\ 0 & a=b,\ b=c,\ a=c \end{cases}$$

$$[\sigma_x, \sigma_y] = 2i\, \sigma_z\, \varepsilon_{xyz} = 2i\, \sigma_z$$

$$[\sigma_y, \sigma_z] = 2i\, \sigma_x$$

$$[\sigma_x, \sigma_x] = 0 \leftarrow$$

- $\{\sigma_a, \sigma_b\} = 2\, \delta_{ab} = \sigma_a \sigma_b + \sigma_b \sigma_a$: Anti commutator

$\Rightarrow$ Identify σ, σ^+ : annihilation and creation op.

$|-\rangle$: ground state

$|+\rangle$: excited state $\qquad E_{|+\rangle} > E_{|-\rangle}$

$$\sigma = |-\rangle\langle+| \qquad \sigma^+ = |+\rangle\langle-|$$

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \langle-| = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

$$|+\rangle\langle-| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \sigma_x + i\sigma_y = \sigma^+$$

$$|-\rangle\langle+| = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \sigma_x - i\sigma_y = \sigma$$

Verify

Postulates of Q.M.

$|\psi\rangle = \alpha|-\rangle + \beta|+\rangle$

$|\pm\rangle$ are the eigenstates of σ_z

eigenvalues are ± 1.

σ_z on $|\psi\rangle$: $\quad |\alpha|^2 : -1 \rightarrow |-\rangle$

$\qquad\qquad\quad |\beta|^2 : +1 \rightarrow |+\rangle$

- What happens if we measure σ_x or σ_y?
  - eigenvectors and eigenvalues of σ_x and σ_y.

For σ_x:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad |\sigma_x - \lambda| = 0$$

$$\det \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - 1 = 0$$

$$(\lambda+1)(\lambda-1) = 0$$

$$\boxed{\lambda_\pm = \pm 1}$$

$$|x_\pm\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \left(|+\rangle \pm |-\rangle \right)$$

Find $|y_\pm\rangle$ and their corresponding eigenvalues

$$|\pm\rangle = \frac{1}{\sqrt{2}} \left(|x_+\rangle \pm |x_-\rangle \right)$$

$$|\psi\rangle = \alpha |-\rangle + \beta |+\rangle$$

- Measure σ_z: find $+1 \to |+\rangle$ with prob. $|\beta|^2$.
- Measure σ_z: find $+1 \to |+\rangle$ with prob. 1.
- Measure σ_x: find $+1 \to |x_+\rangle$ with prob. 1/2.
  find $-1 \to |x_-\rangle$ with prob. 1/2.
  $|x_-\rangle = \frac{1}{\sqrt{2}} \left(|+\rangle - |-\rangle \right)$
- Measure σ_z: find $+1 \to |+\rangle$ with prob. 1/2
  find $-1 \to |-\rangle$ with prob. 1/2