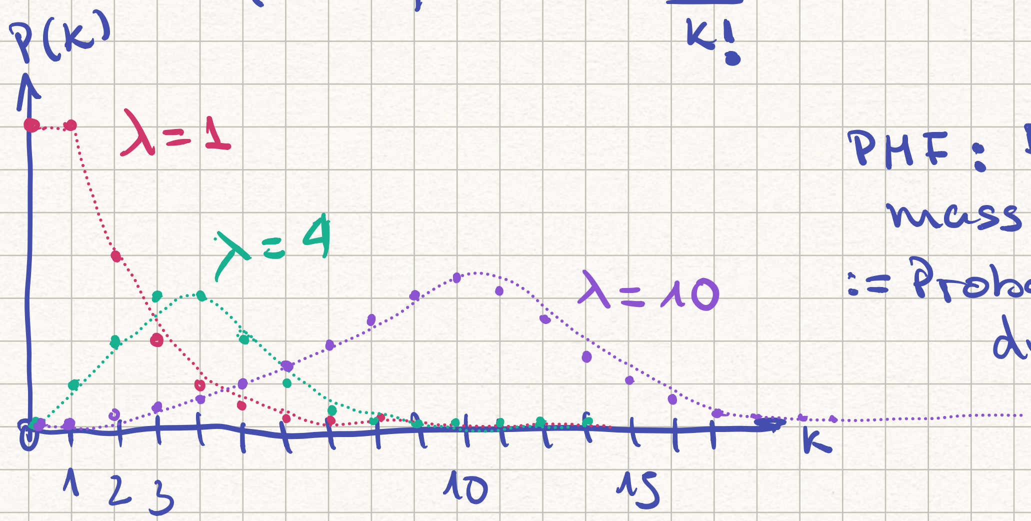
Continuous distributions

Poisson distribution

$p \ll 1$ and $n \gg 1$ $\qquad \lambda = np$: finite

Prob. distribution

$$P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}$$



PMF : Probability mass function := Probability distribution

$\langle X \rangle = \lambda$

Let's consider the case $\lambda = 100$

$$P(X = k) = e^{-100} \frac{100^k}{k!}$$

What's the probability that $X = \lambda$

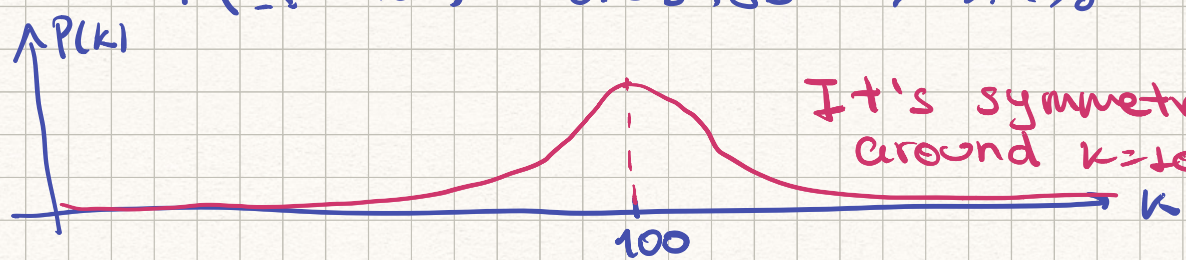
$$P(X = 100) = e^{-100} \frac{100^{100}}{100!} \leq 1$$

$100! = 100 \cdot 99 \cdot 98 \cdot 97 \cdots 1 = ?$ Very large numbers

$100^{100} = ?$

$e^{-100} \cong 3.72 \times 10^{-44}$

$$P(X = 100) \cong 0.03986 \Rightarrow 3.9\%$$



It's symmetric around $k = 100$

We could fit this figure with a simpler function => Gaussian or Normal distribution

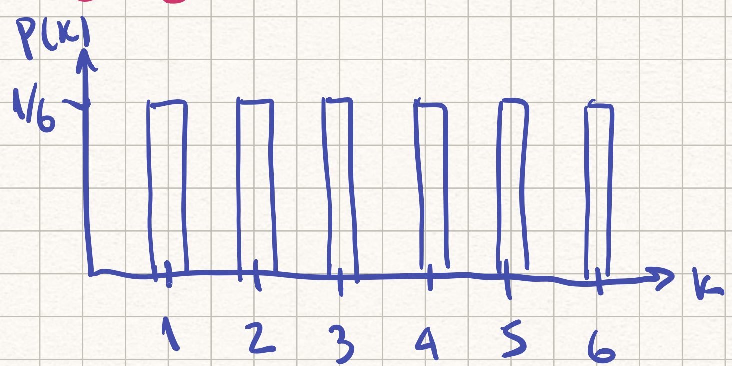
$$f(k) = \frac{1}{\sqrt{2\pi}\,\mu} \exp\left[-\frac{(k-\mu)^2}{2\mu}\right]$$

$\exp[x] = e^x$

If we select $\mu = \lambda$

$$f(100) = \frac{1}{10\sqrt{2\pi}} \approx 0.03989 \rightarrow 3.98\%$$

Playing with dices



with 2 dices

$1+1=2 \rightarrow 6+6=12$

with 3 dices

$1+1+1=3 \rightarrow 6+6+6=18$

The more dices we throw, the smoother the histogram (so, the P(k)) will be

In the limit of infinite dices, the distribution becomes exactly a Gaussian. It even works even if each random variable is not uniform, as long as they are independent

Central limit theorem: the sum of independent random variables tends to a Gaussian

In general, the Gaussian is given by

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

1. The mean, μ
2. The variance, σ^2

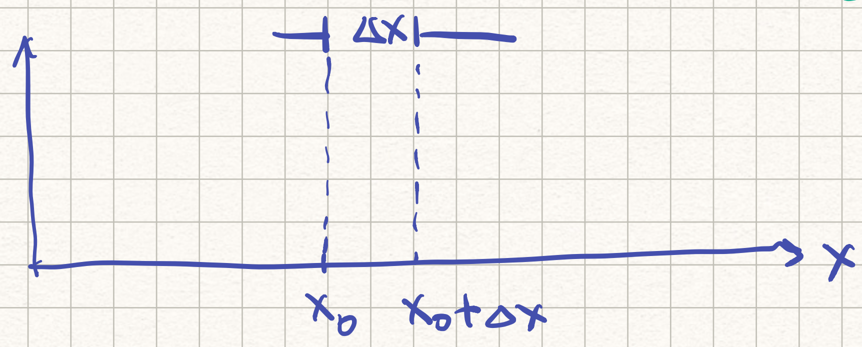
σ : standard deviation

Interpret $f(x)$ as a density of probability for a continuous variable

Now $X \in \mathbb{R}$; is a continuous variable.

$f(x)$ is not a probability $\Rightarrow f(x_0) > 1$

What's the probability of finding $x_0 \leq X \leq x_0 + \Delta x$



$P(x_0) = f(x_0)\Delta x$: Probability that the continuous variable X will return a value between x_0 and $x_0 + \Delta x$

We can only ask the probability to get a number in a window

For $\Delta x \to dx$: differential of $x \to dx \ll 1$

$P(x_0) = f(x_0)dx$

$$P(x_0 \le X \le x_1) = \int_{x_0}^{x_1} f(x)\,dx$$ we are adding probabilities $f(x)\,dx$

Normalization $$\boxed{\int_{-\infty}^{+\infty} f(x)\,dx = 1}$$

The probability to find $-\infty < X < \infty$ is 100%

Averages:

$$\langle X \rangle = \sum_k{}' \, k\, P(k) \quad : \text{discrete variable}$$

$$\langle X \rangle = \int_{-\infty}^{+\infty} x\, f(x)\,dx \quad : \text{continuous variable}$$

Variance

$$\langle X^2 \rangle = \int_{-\infty}^{+\infty} x^2 f(x)\,dx$$

In general $g(X)$: a function of X

$$\boxed{\langle g(X) \rangle = \int_{-\infty}^{+\infty} g(x)\, f(x)\,dx}$$