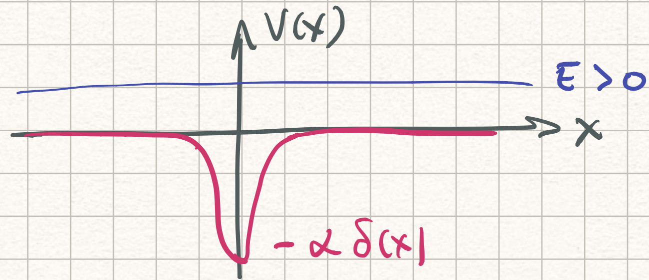
Scattering states of the δ-potential

$\underline{\underline{E>0}}$: scattering states $\Leftrightarrow$ not bounded

$V(x) = -\alpha\,\delta(x)$



$$\left[\frac{p^2}{2m} + V(x)\right]\psi(x) = E\,\psi(x)$$

$$\left[-\frac{\hbar^2\partial_x^2}{2m} - \alpha\,\delta(x)\right]\psi(x) = E\,\psi(x) \Rightarrow \partial_x^2\psi + \frac{2m\alpha}{\hbar^2}\,\delta(x)\,\psi(x) = -\frac{2mE}{\hbar^2}\psi(x)$$

$$= -k^2\psi(x)$$

$$k = \sqrt{\frac{2mE}{\hbar^2}} \in \mathbb{R}$$

Beyond the singularity ($\delta(x)$), except at $x=0$:

$$\partial_x^2\psi = -k^2\psi(x) \Rightarrow \psi(x) = \begin{cases} A e^{ikx} + B e^{-ikx} & x<0 \\ C e^{ikx} + D e^{-ikx} & x>0 \end{cases}$$

$$\psi(x=0) = \left\{\begin{matrix} A+B \\ C+D \end{matrix}\right\} \Rightarrow \boxed{A+B = C+D}$$

$$\int_{-\epsilon}^{\epsilon}\partial_x^2\psi\,dx + \frac{2m\alpha}{\hbar^2}\int_{-\epsilon}^{\epsilon}\delta(x)\,\psi(x)\,dx = -\frac{2mE}{\hbar^2}\int_{-\epsilon}^{\epsilon}\psi(x)\,dx$$

Limit $\epsilon \to 0$

$$\psi'(\epsilon) - \psi'(-\epsilon) + \frac{2m\alpha}{\hbar^2}\psi(0) = 0 \quad ⊗$$

$$\psi'(x) = \begin{cases} ik(Ae^{ikx} - Be^{-ikx}) & x<0 \\ ik(Ce^{ikx} - De^{-ikx}) & x>0 \end{cases}$$

$$\psi'(\epsilon) = ik(Ce^{ik\epsilon} - De^{-ik\epsilon}) = ik(C-D)$$

$$\psi'(-\epsilon) = ik(Ae^{-ik\epsilon} - Be^{ik\epsilon}) = ik(A-B)$$

⊗ $ik(C-D) - ik(A-B) = \frac{-2m\alpha}{\hbar^2}\psi(0)$

$ik(C-D-A+B) = -\frac{2m\alpha}{\hbar^2}(A+B)$

$C-D-A+B = -\frac{2m\alpha}{ik\hbar^2}(A+B)$

$\boxed{-\frac{1}{i} = i}$ Check

$\beta = \frac{m\alpha}{\hbar^2 k}$

$C-D-A+B = 2i\beta(A+B)$

$C-D = A-B+2i\beta(A+B)$

$\boxed{C-D = A(1+2i\beta) - B(1-2i\beta)}$

$\boxed{A+B = C+D}$

4 unknowns + K
we only have 2 eqs.!!!

Normalization doesn't help ⇒ not normalizable!

$Ae^{ikx} + Be^{-ikx}$ Try it!

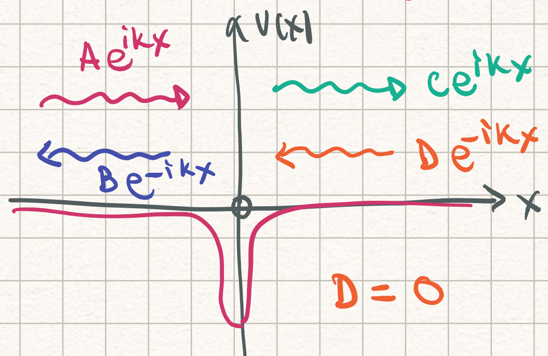
$\boxed{\text{Physical interpretation}}$

$\psi(x,t) = \psi(x)e^{-iEt/\hbar}$

$= Ae^{i(kx-Et/\hbar)} + Be^{-i(kx+Et/\hbar)}$

wave that propagates to the right

waves that propagate to the left



$D = 0$

→ Scattering of particle that comes from the left ⟶

→ Ae^{ikx}: Incoming

→ Ce^{ikx}: Transmitted

→ Be^{-ikx}: Reflected

$C = A(1+2i\beta) - B(1-2i\beta)$

$C = A+B$

$A(1+2i\beta) - B(1-2i\beta) = A+B$

$A(1+2i\beta) - A = B + B(1-2i\beta)$

$2i\beta A = 2B(1-i\beta)$

$\boxed{B = \frac{i\beta}{1-i\beta}A}$

$$C = A + B = A + \frac{i\beta}{1-i\beta}A = A\left[1 + \frac{i\beta}{1-i\beta}\right] = \boxed{A\,\frac{1}{1-i\beta} = C}$$

Scattering from the RIGHT, set $A = 0$ and then

$$B = \frac{1}{1-i\beta}\underline{D} \qquad C = \frac{i\beta}{1-i\beta}\underline{D} \qquad \leftarrow De^{-ikx}$$

$|\Psi|^2$: density of prob x in space

Relative probability that the particle is

Reflected — Transmitted

$$R \equiv \frac{|B|^2}{|A|^2} = \frac{|A|^2}{|A|^2}\left|\frac{i\beta}{1-i\beta}\right|^2 \qquad T \equiv \frac{|C|^2}{|A|^2} = \frac{|A|^2}{|A|^2}\left|\frac{1}{1-i\beta}\right|$$

Reflection coefficient — Transmission coefficient

$$\left|\frac{i\beta}{1-i\beta}\right|^2 = \left(\frac{i\beta}{1-i\beta}\right)\left(\frac{-i\beta}{1+i\beta}\right) = \frac{\beta^2}{1+\beta^2} \qquad \left|\frac{1}{1-i\beta}\right|^2 = \frac{1}{1+\beta^2}$$

$$R = \frac{\beta^2}{1+\beta^2} \qquad T = \frac{1}{1+\beta^2}$$

Of course $\boxed{R + T = 1}$

$$\beta = \frac{m\alpha}{\hbar^2 k} \quad \text{and} \quad k = \frac{\sqrt{2mE}}{\hbar}$$

$$= \frac{m\alpha}{\hbar\sqrt{2mE}} \Rightarrow \beta^2 = \frac{m^2\alpha^2}{\hbar^2\, 2mE} = \frac{m\alpha^2}{2E\hbar^2}$$

$$R = \frac{\beta^2}{1+\beta^2} = \frac{\beta^2}{\beta^2(1/\beta^2 + 1)} = \frac{1}{1 + 1/\beta^2} \qquad 1 + 1/\beta^2 = 1 + \frac{2E\hbar^2}{m\alpha^2}$$

$$T = \frac{1}{1+\beta^2} \qquad 1 + \beta^2 = 1 + \frac{m\alpha^2}{2E\hbar^2}$$

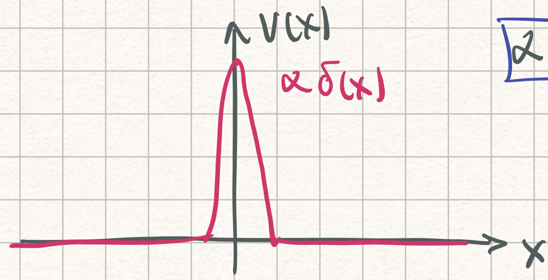
$$R = \frac{1}{1 + 2E\hbar^2/(m\alpha^2)} \qquad T = \frac{1}{1 + \underline{\underline{m\alpha^2/(2E\hbar^2)}}}$$

when $E \gg 1$: $R \to 0$

$T \to 1$

Combination of stationary states → wavepacket
↳ range of energies

R and T are approximates for a particle with an energy in the vicinity of E.



$\boxed{\alpha \to -\alpha}$

- There's no bound state

{ Check what happens when $\alpha \to -\alpha$ in the solutions of the previous lecture! }

R and T are functions of α^2 ⇒ they remain the same

The particle is as likely to pass (transmitted) over a well than through a barrier!!!

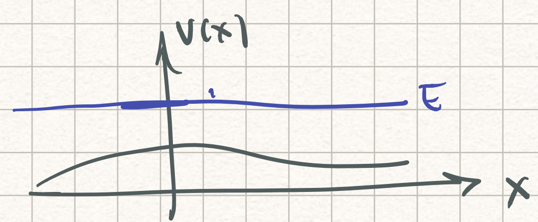
Classically

If $E > V_{max}$:

$R = 0$; $T = 1$

$E < V_{max}$

$R = 1$; $T = 0$



Quantum world

nonzero prob. go through $E < V_{max}$

→ Quantum tunneling !!!

$E > V_{max}$: non zero prob. the particle will be reflected back.