Random Variables

Deterministic objects

→ variable x : fixed in time

x: # of patients at hospital

$x(t)$: # of patiens depending on t

Non-deterministic objects

→ random variable X : outcome of a dice

Throwing a dice 8 times:

$X = 1, 3, 4, 6, 5, 3, 1, 2$

Probabilities remain constant

$p(n)$: probability of getting n as an outcome

$$p(1) = p(2) = p(3) = p(4) = p(5) = p(6) = 1/6$$

$$\sum_{n=1}^{6} p(n) = p(1) + p(2) + \cdots + p(6) = 1$$ Probability is normalized

normalized: sum of all the probabilities is 1

If $p(n)$ is the same for every n ⇒ Uniform distribution

Tossing a coin: $\begin{cases} \text{Heads } (H) \Rightarrow -1 \\ \text{Tails } (T) \Rightarrow 1 \end{cases}$

$p(H) = p(T) = 1/2$ $\quad$ $p(H) + p(T) = 1$

$p(-1) = p(1) = 1/2$ $\quad$ $p(-1) + p(1) = 1$

Uniform distribution

For a biased coin: $p(-1) = 0.6$ thus $p(1) = 0.4$

The prob. distributions MUST be normalized

Dice: $\sum_{n=1}^{6} p(n) = 1$

This is true regardless of the particular values of $p(n)$.

Coin: $\sum_{n=-1,1} p(n) = 1$

In general: $\boxed{\sum_{n} p(n) = 1}$

Average

$\langle\ \rangle$: statistical average (mean)

$\langle X \rangle$: statistical average of the random variable X.

Dice:

n	$p(n)$
1	1/6
2	1/6
3	1/6
4	1/6
5	1/6
6	1/6

$$\langle X \rangle = \sum_{n=1}^{6} n\, p(n)$$

$$= 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + \cdots + 6 \cdot \frac{1}{6}$$

$$= \frac{7}{2} = 3.5$$

Coin:

n	$p(n)$
-1	1/2
1	1/2

$$\langle X \rangle = \sum_{n=-1,1} n\, p(n) = 0$$

What is $\langle X \rangle$ if $p(-1) = 0.6$?

Multiple random variables

X : one dice $\qquad$ Y : another dice

$P_X(n) = 1/6$ for $n=1\dots,6$ $\qquad$ $P_Y(n) = 1/6$ for $n=1,\dots,6$

They will provide different values everytime we throw them:

$$(X, Y) = (1,3), (2,5), (6,6), (1,4) \dots$$

In general the distribution for X and Y are different.

Sometimes, you are interested in $Z = X + Y$

$$Z = 4, 7, 12, 5 \dots$$

Z is a random variable which is a function of two random variables

$\langle Z \rangle = ?$

Average is a linear application

$$\langle \alpha X + \beta Y \rangle = \alpha \langle X \rangle + \beta \langle Y \rangle \qquad \alpha, \beta \in \mathbb{C}$$

Two fair dices $\langle X \rangle = \langle Y \rangle = 3.5 \Rightarrow \langle Z \rangle = 7.$

X and Y were independent

↳ the outcome of X is independent from the outcome of Y

↳ the two variables are uncorrelated

Let's consider $Y = 7 - X$ Perfect correlation

Y: opposite side of the dice

$X = 1, 6, 3, 5, 2, 3, 4$

$Y = 6, 1, 4, 2, 5, 4, 3$

Functions of random variables

For a deterministic variable $f(x) = x^2$

For a random variable $Y = f(X)$

$Y = X^2$

Coin:

$X = -1, 1, 1, 1, -1, -1, 1, 1$

$Y = 1, 1, 1, 1, 1, 1, 1, 1$

For Y: $P_Y(n) = \delta_{n,1}$ $\delta_{n,1} = \begin{cases} 1 & n = 1 \\ 0 & \text{otherwide} \end{cases}$

Dice:

$X = 1, 3, 5, 2, 1, 1, 3, 6$

$Y = 1, 9, 25, 4, 1, 1, 9, 36$

What is probability that $Y = 7$?

$$\langle Y \rangle = \langle X^2 \rangle = \sum_n n^2 p(n)$$

Show that for the fair dice $\langle X^2 \rangle = \frac{91}{6}$

Note: $\langle X^2 \rangle \neq \langle X \rangle^2$ $\langle X^2 \rangle \geq \langle X \rangle^2$

Dice: $\langle X^2 \rangle = 91/6 \approx 15.16$

$\langle X \rangle^2 = (7/2)^2 \approx 12.25$

$$\mathrm{Var}(\underline{X}) \equiv \langle \underline{X}^2 \rangle - \langle \underline{X} \rangle^2 \geq 0$$: Variance of $\underline{X}$

- The distance from $\underline{X}$ to the mean value $\langle \underline{X} \rangle$

  $\underline{X} - \langle \underline{X} \rangle$: another random number
- $(\underline{X} - \langle \underline{X} \rangle)^2$: another random number

$$\mathrm{Var}(\underline{X}) = \langle (\underline{X} - \langle \underline{X} \rangle)^2 \rangle$$

- $(\underline{X} - \langle \underline{X} \rangle)^2 = \underline{X}^2 - 2\langle \underline{X} \rangle \underline{X} + \langle \underline{X} \rangle^2$

$$\begin{aligned} \langle (\underline{X} - \langle \underline{X} \rangle)^2 \rangle &= \langle \underline{X}^2 - 2\langle \underline{X} \rangle \underline{X} + \langle \underline{X} \rangle^2 \rangle \\ &= \langle \underline{X}^2 \rangle - 2\langle \underline{X} \rangle \langle \underline{X} \rangle + \langle \underline{X} \rangle^2 \\ &= \langle \underline{X}^2 \rangle - 2\langle \underline{X} \rangle^2 + \langle \underline{X} \rangle^2 \\ &= \langle \underline{X}^2 \rangle - \langle \underline{X} \rangle^2 \end{aligned}$$

standard deviation $\sigma_{\underline{X}} = \sqrt{\mathrm{Var}(\underline{X})}$

You can also obtain $\langle \underline{X}^3 \rangle \Rightarrow$

In general $f(\underline{X}) = \underline{Y}$

$$\langle \underline{Y} \rangle = \langle f(\underline{X}) \rangle = \sum_n{}' f(n)\, p(n)$$

$$f(\underline{X}) = e^{-\underline{X}} \Rightarrow \langle f(\underline{X}) \rangle = \sum_n{}' e^{-n}\, p(n)$$

$$f(\underline{X}) = \tan(\underline{X}) \Rightarrow \langle f(\underline{X}) \rangle = \sum_n{}' \tan(n)\, p(n)$$