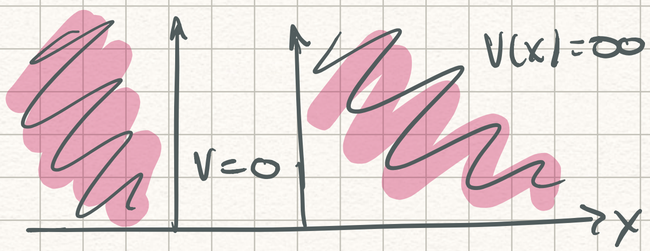
Particle in a box

$\psi(x) \sim \sin(kx)$

$E < V$

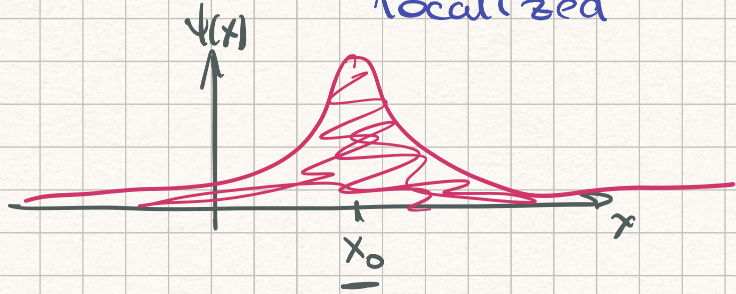


Free particle

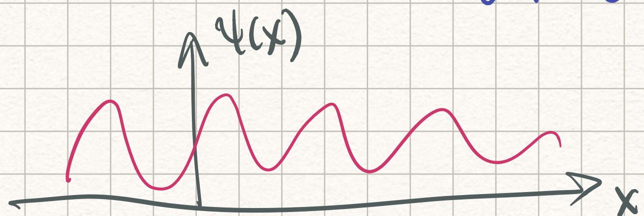
$\psi(x) \propto e^{i(kx-\omega t)}$

$E > V \quad (V=0)$

- Bound state: particle is localized



- Scattering state: how a potential affects a particle without bounding it.

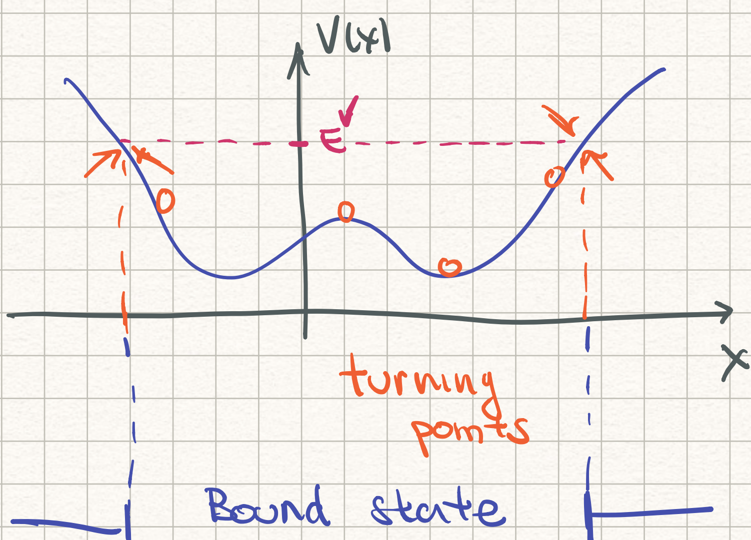


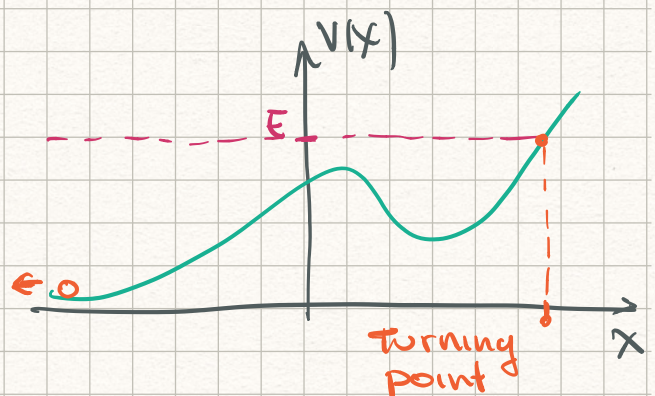
Bound state $E < V(\pm\infty)$ typically $\underline{\underline{V(\pm\infty) = 0}}$

Scattering state $E > V(\pm\infty)$

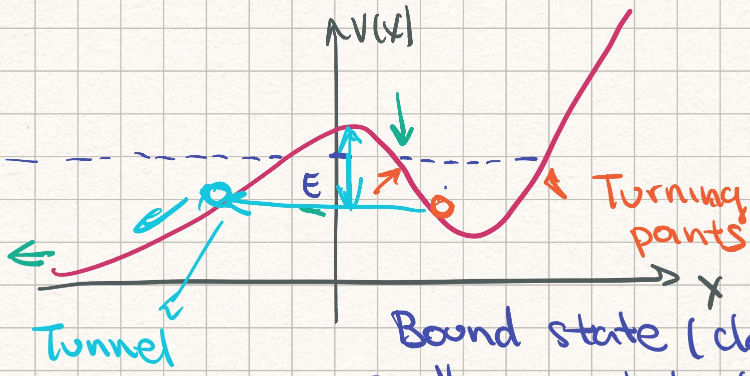
Bound state $E < 0$

Scattering state $E > 0$





Scattering states



Bound state (classical)
Scattering state (quantum)

Quantum Tunneling!

Delta Potential: $V(x) = -\alpha\,\delta(x)$

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases}$$

- generalized function
- distribution

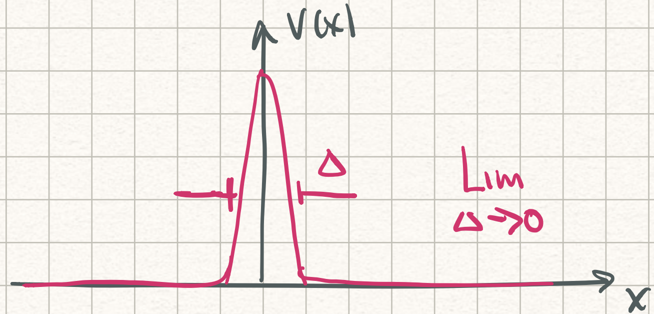
$$\boxed{\int_{-\infty}^{\infty} \delta(x)\,dx = 1}$$

$$f(x)\,\delta(x-a) = f(a)\,\delta(x-a)$$

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = \int_{-\infty}^{\infty} f(a)\,\delta(x-a)\,dx$$

$$= f(a)\int_{-\infty}^{\infty} \delta(x-a)\,dx$$

$$= f(a)$$



$\lim_{\Delta \to 0}$

- What are the dimensions of the constant α?

Write me an email with your answer!

$$i\hbar \frac{\partial \psi}{\partial t} = \left[\frac{p^2}{2m} + V(x)\right]\psi(x,t)$$

$$\left(\frac{p^2}{2m} + V(x)\right)\psi(x) = E\,\psi(x) \qquad p \to -i\hbar\partial_x$$

$$\left[-\frac{\hbar^2 \partial_x^2}{2m} - \alpha\,\delta(x)\right]\psi(x) = E\,\psi(x)$$

$$\left[\frac{\hbar^2 \partial_x^2}{2m} + \alpha\,\delta(x)\right]\psi(x) = -E\,\psi(x)$$

Let's look for a bound state $(E<0)$

$$V(x) = -\alpha\,\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases}$$

$\boxed{x<0}$ $V(x)=0$

$$\frac{\hbar^2 \partial_x^2 \psi}{2m} = -E\psi(x) \Rightarrow \partial_x^2\psi = -\frac{2mE}{\hbar^2}\psi$$
$$= k^2\psi$$

$$\boxed{k^2 = -\frac{2mE}{\hbar^2}} > 0 \quad E<0 \qquad k, \text{ is Real and } k>0$$

$$\partial_x^2\psi = k^2\psi \Rightarrow \psi(x) = Ae^{kx} + Be^{-kx}$$

$$\lim_{x\to-\infty}\psi(x) = \lim_{x\to-\infty} Be^{-kx} \Rightarrow B=0 \qquad \boxed{\psi(x) = Ae^{kx}}, x<0$$

$\boxed{x>0,\ V(x)=0}$

$$\frac{\hbar^2 \partial_x^2 \psi}{2m} = -E\psi(x) \Rightarrow \partial_x^2\psi = -\frac{2mE}{\hbar^2}\psi$$
$$= k^2\psi$$

$$k^2 = -\frac{2mE}{\hbar^2} > 0 \quad E<0 \qquad k, \text{ is Real and } k>0$$

$$\partial_x^2\psi = k^2\psi \Rightarrow \psi(x) = Ce^{kx} + De^{-kx}$$

$$\lim_{x\to\infty}\psi(x) = \lim_{x\to\infty} Ce^{kx} = 0 \Rightarrow C=0 \qquad \boxed{\psi(x) = De^{-kx}} \quad x<0$$

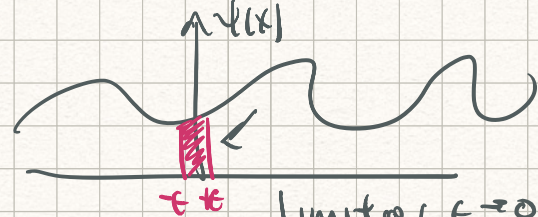
$$\psi(x) = \begin{cases} Ae^{kx} & x<0 \\ De^{-kx} & x>0 \end{cases}$$

1. $\psi(x)$ has to be continuous everywhere
2. $\frac{d\psi}{dx}$ has to be continuous, except when $V(x)\to\infty$

$$\psi(x=0) = A = D \qquad \psi(x) = \begin{cases} Ae^{kx} & x<0 \\ Ae^{-kx} & x>0 \end{cases}$$

$$-\frac{\hbar^2}{2m}\underbrace{\int_{-\epsilon}^{\epsilon}\frac{d^2\psi}{dx^2}dx}_{\psi'(x)} - \int_{-\epsilon}^{\epsilon}\alpha\,\delta(x)\,\psi(x)\,dx = E\int_{-\epsilon}^{\epsilon}\psi(x)\,dx$$

Take limit $\epsilon \to 0$



Limit of $\epsilon \to 0$

$$-\frac{\hbar^2}{2m}\left[\psi'(\epsilon)-\psi'(-\epsilon)\right] - \alpha\psi(0) = 0$$

• $\epsilon \to 0$

$$\psi'(x) = \begin{cases} Ak\,e^{kx} & x<0 \\ -Ak\,e^{-kx} & x>0 \end{cases}$$

$$\lim_{\epsilon\to 0}\left[\psi'(\epsilon)-\psi'(-\epsilon)\right] = \lim_{\epsilon\to 0}\left[-Ak\,e^{-k\epsilon} - Ak\,e^{-k\epsilon}\right]$$

$$= -2Ak\lim_{\epsilon\to 0} e^{-k\epsilon} = -2Ak$$

$$-\frac{\hbar^2}{2m}(-2Ak) - \alpha\psi(0) = 0$$

$$\frac{\hbar^2 Ak}{m} = \alpha\psi(0) = \alpha A \Rightarrow \boxed{k = \frac{m\alpha}{\hbar^2}}$$

$$k^2 = -\frac{2mE}{\hbar^2} = \left(\frac{m\alpha}{\hbar^2}\right)^2 = \frac{m^2\alpha^2}{\hbar^4} \Rightarrow E = -\frac{\hbar^2}{2m}\frac{m^2\alpha^2}{\hbar^4}$$

$$\boxed{E = -\frac{m\alpha^2}{2\hbar^2}}$$

• Is this expression consistent with your previous answer?
$[\alpha] = ?$

Send this answer also through email!

$$\psi(x) = \begin{cases} A\,e^{kx} & x<0 \\ A\,e^{-kx} & x>0 \end{cases}$$

Find A by normalization

$$1 = \int_{-\infty}^{\infty}|\psi(x)|^2dx = \int_{-\infty}^{0}|\psi(x)|^2dx + \int_{0}^{\infty}|\psi(x)|^2dx$$

$$= |A|^2\int_{-\infty}^{0}e^{2kx}dx + |A|^2\int_{0}^{\infty}e^{-2kx}dx$$

$$= |A|^2 \left[\frac{1}{2k} + \frac{(-1)}{(-2k)} \right] = \frac{|A|^2}{k} = 1 \quad \Rightarrow |A| = \sqrt{k}$$

$$\psi(x) = \begin{cases} \sqrt{k}\, e^{kx} & x < 0 \\ \sqrt{k}\, e^{-kx} & x > 0 \end{cases}$$

If $x < 0$: $|x| = -x$

$|-5| = 5 = -(-5)$

$x = -|x|$

$e^{kx} = e^{-k|x|}$

If $x > 0$: $|x| = x$

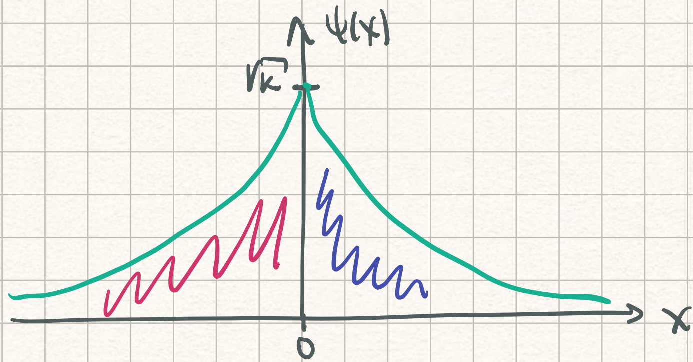
$|5| = 5 = 5$

$-x = -|x|$

$e^{-kx} = e^{-k|x|}$

$\psi(x) = \sqrt{k}\, e^{-k|x|}$ $\qquad k = \frac{m\alpha}{\hbar^2}$

$$\boxed{\psi(x) = \frac{\sqrt{m\alpha}}{\hbar}\, e^{-m\alpha|x|/\hbar^2} \qquad E = -\frac{m\alpha^2}{2\hbar^2}}$$



- By simmetry $\langle x \rangle = 0$ Check!

$$\int_{-\infty}^{\infty} \psi^* x \psi \, dx = 0$$

$\langle x^2 \rangle$, $\langle p \rangle$, $\langle p^2 \rangle$

σ_x^2 $\quad \sigma_p^2$ $\quad \sigma_x \sigma_p$

Home works. Please them by email.