# Spontaneous emission

$$\partial_t \rho = \frac{i}{\hbar}[\rho, H] + \frac{1}{2}\sum_k \mathcal{L}_{L_k}\rho$$

$$\mathcal{L}_{L_k}\rho = 2L_k\rho L_k^\dagger - L_k^\dagger L_k \rho - \rho L_k^\dagger L_k$$

The master eq. can always be written as

$$\partial_t \tilde{\rho} = L\tilde{\rho} \qquad L\text{: Liouvillian matrix}$$

$\tilde{\rho}$: is a "vectorized" representation of $\rho$

For a 2LS

$$\rho = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \Rightarrow \tilde{\rho} = \begin{pmatrix} \rho_{00} \\ \rho_{01} \\ \rho_{10} \\ \rho_{11} \end{pmatrix}$$

The solution to the master eq.

$$\boxed{\tilde{\rho}(t) = e^{Lt}\tilde{\rho}(0)}$$

Let's consider : 2LS + $\gamma_\sigma$ + $P_\sigma$

$$H = \hbar\omega_\sigma \sigma^\dagger\sigma$$

$$L_1 = \sqrt{\gamma_\sigma}\,\sigma \qquad L_2 = \sqrt{P_\sigma}\,\sigma^\dagger$$

$$\sigma = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \sigma^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \Rightarrow \sigma^\dagger\sigma = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$H = \begin{pmatrix} 0 & 0 \\ 0 & \hbar\omega_\sigma \end{pmatrix}$$

⊗ For $L_1 = \sqrt{\gamma_\sigma}\,\sigma$ $\qquad 2L_1\rho L_1^\dagger - L_1^\dagger L_1\rho - \rho L_1^\dagger L_1$

$$L_1 \rho L_1^\dagger = \gamma_\sigma \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$= \gamma_\sigma \begin{pmatrix} \rho_{11} & 0 \\ 0 & 0 \end{pmatrix}$$

$$L_1^\dagger L_1 \rho = \gamma_\sigma \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} = \gamma_\sigma \begin{pmatrix} 0 & 0 \\ \rho_{10} & \rho_{11} \end{pmatrix}$$

$$\rho L_1^\dagger L_1 = \gamma_\sigma \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \gamma_\sigma \begin{pmatrix} 0 & \rho_{01} \\ 0 & \rho_{11} \end{pmatrix}$$

$$2 L_1 \rho L_1^\dagger - L_1^\dagger L_1 \rho - \rho L_1^\dagger L_1 = \gamma_\sigma \begin{pmatrix} 2\rho_{11} & -\rho_{01} \\ -\rho_{10} & -2\rho_{11} \end{pmatrix}$$

$\otimes$ $[\rho, H]$

$$H = \begin{pmatrix} 0 & 0 \\ 0 & \hbar\omega_\sigma \end{pmatrix} \qquad \rho H = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \hbar\omega_\sigma \end{pmatrix} = \begin{pmatrix} 0 & \hbar\omega_\sigma \rho_{01} \\ 0 & \hbar\omega_\sigma \rho_{11} \end{pmatrix}$$

$$H\rho = \begin{pmatrix} 0 & 0 \\ 0 & \hbar\omega_\sigma \end{pmatrix} \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \hbar\omega_\sigma \rho_{10} & \hbar\omega_\sigma \rho_{11} \end{pmatrix}$$

$$\rho H - H\rho = \begin{pmatrix} 0 & \hbar\omega_\sigma \rho_{01} \\ 0 & \hbar\omega_\sigma \rho_{11} \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ \hbar\omega_\sigma \rho_{10} & \hbar\omega_\sigma \rho_{11} \end{pmatrix}$$

$$= \hbar\omega_\sigma \begin{pmatrix} 0 & \rho_{01} \\ -\rho_{10} & 0 \end{pmatrix}$$

$\otimes$ $L_2 = \sqrt{P_\sigma}\, \sigma^+$

$$2 L_2 \rho L_2^\dagger - L_2^\dagger L_2 \rho - \rho L_2^\dagger L_2 = P_\sigma \begin{pmatrix} -2\rho_{00} & -\rho_{01} \\ -\rho_{10} & 2\rho_{00} \end{pmatrix}$$ check on your own!

Putting everything together

$$\partial_t \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} = \frac{i}{\hbar} \hbar\omega_\sigma \begin{pmatrix} 0 & \rho_{01} \\ -\rho_{10} & 0 \end{pmatrix} + \frac{\gamma_\sigma}{2} \begin{pmatrix} 2\rho_{11} & -\rho_{01} \\ -\rho_{10} & -2\rho_{11} \end{pmatrix}$$

$$+\frac{P_\sigma}{2}\begin{pmatrix} -2\rho_{00} & -\rho_{01} \\ -\rho_{10} & 2\rho_{00} \end{pmatrix}$$

$$\partial_t \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} = \begin{bmatrix} \gamma_\sigma \rho_{11} - P_\sigma \rho_{00} & \left(i\omega_\sigma - \frac{\gamma_\sigma + P_\sigma}{2}\right)\rho_{01} \\ \left(i\omega_\sigma - \frac{\gamma_\sigma + P_\sigma}{2}\right)\rho_{10} & P_\sigma \rho_{00} - \gamma_\sigma \rho_{11} \end{bmatrix}$$

$$\partial_t \begin{bmatrix} \rho_{00} \\ \rho_{01} \\ \rho_{10} \\ \rho_{11} \end{bmatrix} = \begin{bmatrix} -P_\sigma & 0 & 0 & \gamma_\sigma \\ 0 & \left(i\omega_\sigma - \frac{\gamma_\sigma + P_\sigma}{2}\right) & 0 & 0 \\ 0 & 0 & \left(i\omega_\sigma - \frac{\gamma_\sigma + P_\sigma}{2}\right) & 0 \\ P_\sigma & 0 & 0 & -\gamma_\sigma \end{bmatrix} \begin{bmatrix} \rho_{00} \\ \rho_{01} \\ \rho_{10} \\ \rho_{11} \end{bmatrix}$$

$$L = \begin{bmatrix} -P_\sigma & 0 & 0 & \gamma_\sigma \\ 0 & \left(i\omega_\sigma - \frac{\gamma_\sigma + P_\sigma}{2}\right) & 0 & 0 \\ 0 & 0 & \left(i\omega_\sigma - \frac{\gamma_\sigma + P_\sigma}{2}\right) & 0 \\ P_\sigma & 0 & 0 & -\gamma_\sigma \end{bmatrix}$$ : Liouvillian

$$\tilde{\rho} = e^{Lt}\tilde{\rho}$$

Find $\exp(Lt)$

- Diagonalization

Spontaneous emission of N photons

We need a harmonic oscillator with N photons

- EM-field
- polaritons
- phonons
- plasmos

any bosonic field

$|\psi(t=0)\rangle = |N\rangle$ $\quad$ $|N\rangle$: state with N particles

$H = \hbar\omega_a a^\dagger a$

$a = \sqrt{n}\,|n-1\rangle\langle n|$

$a^\dagger = \sqrt{n+1}\,|n+1\rangle\langle n|$

$a^\dagger a = n\,|n\rangle\langle n|$

$L_1 = \sqrt{\gamma_a}\, a$

The effective Hamiltonian $\tilde{H} = H - \frac{i\hbar}{2} \sum_k L_k^\dagger L_k$

$$\tilde{H} = \hbar\omega_a a^\dagger a - \frac{i\hbar}{2} \gamma_a a^\dagger a$$

$$= \hbar\left(\omega_a - \frac{i\gamma_a}{2}\right) a^\dagger a = \hbar\tilde{\omega}\, a^\dagger a$$

we are interested in $e^{-i\tilde{H}\delta t/\hbar}$

with the 2LS

$$\sigma = \begin{array}{cc} \begin{array}{cc} \langle g| & \langle e| \end{array} & \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{array}{c} |g\rangle \\ |e\rangle \end{array} \end{array} \qquad \sigma^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \Rightarrow \sigma^\dagger\sigma = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$a = \begin{array}{cc} \begin{array}{cccccc} \langle 0| & \langle 1| & \langle 2| & \langle 3| & \cdots & \end{array} & \\ \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \sqrt{2} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \sqrt{3} & & \\ & & & & \sqrt{4} & \\ & & & & & \ddots \end{pmatrix} & \begin{array}{c} |0\rangle \\ |1\rangle \\ |2\rangle \\ |3\rangle \\ \vdots \end{array} \end{array}$$

$a = |n-1\rangle\langle n| \sqrt{n}$

$|1\rangle\langle 0|$ $\quad$ $|2\rangle\langle 0|$

$|1\rangle\langle 1|$ $\quad$ $|2\rangle\langle 1|$

$|1\rangle\langle 2|$ $\quad$ $|2\rangle\langle 2|$

$|1\rangle\langle 3|$ $\quad$ $|2\rangle\langle 3|$

$$a^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 & & \\ 1 & 0 & 0 & 0 & & \\ 0 & \sqrt{2} & 0 & 0 & & \\ \vdots & 0 & \sqrt{3} & 0 & & \\ \vdots & \vdots & \vdots & \sqrt{4} & \ddots & \\ 0 & 0 & 0 & \vdots & & 0 \end{pmatrix}$$

$$a^\dagger a = \begin{pmatrix} 0 & & & & & \\ & 1 & & & 0 & \\ & & 2 & & & \\ & & & 3 & & \\ & 0 & & & 4 & \\ & & & & & \ddots \end{pmatrix} \Rightarrow e^{a^\dagger a} = \begin{pmatrix} e^0 & & & & & \\ & e^1 & & & 0 & \\ & & e^2 & & & \\ & & & e^3 & & \\ & 0 & & & e^4 & \\ & & & & & \ddots \\ & & & & & & e^n \end{pmatrix}$$

$$e^{-i\tilde{H}\delta t/\hbar} = e^{-i(\hbar\tilde{\omega}a^\dagger a)\delta t/\hbar} = e^{-i\tilde{\omega}\delta t\, a^\dagger a}$$

$$= \begin{pmatrix} e^0 & & & & \\ & e^{-i\tilde{\omega}\delta t} & & 0 & \\ & & e^{-2i\tilde{\omega}\delta t} & & \\ & 0 & & e^{-3i\tilde{\omega}\delta t} & \\ & & & & \ddots \end{pmatrix}$$

We start with $|\psi(0)\rangle = |N\rangle$

$\hat{n} = a^\dagger a$

- Hamiltonian evolution

$$|\psi^{(1)}(\delta t)\rangle = e^{-i\tilde{\omega}\delta t\hat{n}}|N\rangle$$

$$= e^{-i\tilde{\omega}\delta t N}|N\rangle \quad \text{check!}$$

$$\tilde{\omega} = \omega_a - i\gamma_a/2$$

$$|\psi^{(1)}(\delta t)\rangle = e^{-i\omega_a\delta t N}\, e^{-\gamma_a\delta t N/2}|N\rangle$$

$$\langle\psi^{(1)}(\delta t)| = e^{i\omega_a\delta t N}\, e^{-\gamma_a\delta t N/2}\langle N|$$

$$\langle\psi^{(1)}(\delta t)|\psi^{(1)}(\delta t)\rangle = e^{-\gamma_a\delta t N}$$

$$\boxed{|\psi(\delta t)\rangle = e^{-i\omega_a\delta t N}|N\rangle}$$

- Quantum jump

$$\delta P_k = \delta t\,\langle\psi(0)|\gamma_a a^\dagger a|\psi(0)\rangle = \delta t\,\gamma_a\langle N|a^\dagger a|N\rangle$$

$$= \delta t\,\gamma_a N$$

Draw $\epsilon$: random number

$$\delta P_k > \epsilon \Rightarrow |\psi^{(2)}(\delta t)\rangle = \sqrt{\gamma_a}\, a|N\rangle$$

$$= \sqrt{\gamma_a}\sqrt{N}\,|N-1\rangle$$

$$\langle \psi^{(2)}(\delta t)| = \sqrt{N\gamma_a}\,\langle N-1|$$

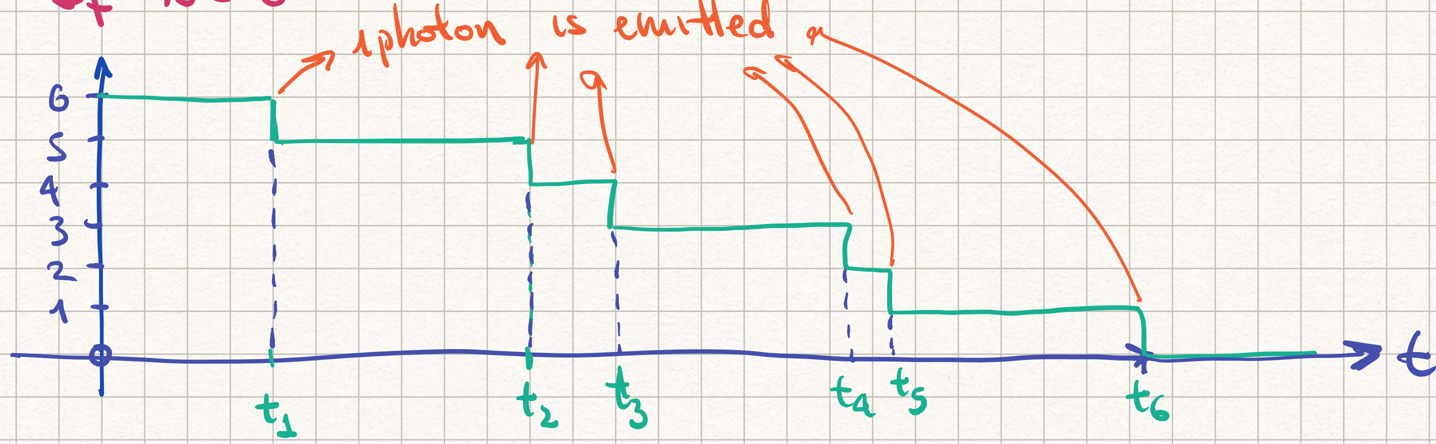
$$\langle \psi^{(2)}(\delta t)|\psi^{(2)}(\delta t)\rangle = \gamma_a N$$

$$|\psi(\delta t)\rangle = |N-1\rangle$$

Performing the MC experiment

$\langle a^\dagger a\rangle(t)$: How many photons are there in $|\psi(t)\rangle$

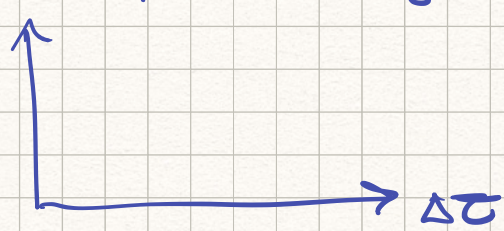
If $N = 6$




⊗ Get 10.000 realizations

⊗ Plot

- Probability of emitting the 1st photon vs time
- Probability that two consecutive photons will be separated by $\Delta\tau$




⊗ what is the mean $\langle t_k\rangle$

- Histogram → distribution