Uncertainty Principle

$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$ $\qquad$ $\sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2}$

Heisenberg $\qquad \underline{\sigma_x \sigma_p} \geq \frac{\hbar}{2}$

- If we know the position exactly : $\langle x\rangle$ is well defined and $\sigma_x \to 0$

$$\sigma_x \sigma_p \geq \frac{\hbar}{2} \Rightarrow \sigma_p \to \infty$$

If we know very well the position $\langle x\rangle$, then we know very little about the momentum $\langle k\rangle$.

$\phi(k)$: very localized $\to$ $\psi(x)$ was unlocalized
$\phi(k)$: very delocalized $\to$ $\psi(x)$ was localized
$\phi(k) = 1$ $\qquad$ $\psi(x) = \sqrt{2\pi}\,\delta(x)$

Demonstration

A, B such $[A,B] = AB - BA \neq 0$ They don't commute

A and B are observables

$$\sigma_A^2 = \langle\psi|\left(A - \langle A\rangle\right)^2|\psi\rangle \quad (\times)$$

$\langle\psi|\left(A^2 - A\langle A\rangle - \langle A\rangle A + \langle A\rangle^2\right)|\psi\rangle$
$= \langle\psi|A^2|\psi\rangle - 2\langle A\rangle\langle A\rangle + \langle A\rangle^2$
$= \langle A^2\rangle - \langle A\rangle^2$

$A^+ = A$
$B^+ = B$

A is hermitian $\to$ $\langle A\rangle \in \mathbb{R}$

$A - \langle A\rangle$ is also hermitian $\Rightarrow$ $\underline{A - \langle A\rangle = (A - \langle A\rangle)^+}$

$$\sigma_A^2 = \langle\psi|(A-\langle A\rangle)^2|\psi\rangle$$

$$= \langle\psi|(A-\langle A\rangle)(A-\langle A\rangle)|\psi\rangle$$

$$= \langle\psi|(A-\langle A\rangle)^{+}(A-\langle A\rangle)|\psi\rangle$$

$$|\alpha\rangle = (A-\langle A\rangle)|\psi\rangle$$

$$\langle\alpha| = \langle\psi|(A-\langle A\rangle)^{+}$$

$$= \langle\alpha|\alpha\rangle$$

$\sigma_B^2 = \langle\beta|\beta\rangle$ using $|\beta\rangle = (B-\langle B\rangle)|\psi\rangle$

$|\alpha\rangle$ and $|\beta\rangle$ are vectors! $\Rightarrow$ vector space

$\rightarrow$ inner product

$$\langle\alpha|\alpha\rangle = ||\alpha\rangle|^2 \qquad \vec{u}\cdot\vec{u} = |\vec{u}|^2$$

$$\langle\beta|\beta\rangle = ||\beta\rangle|^2$$

$$\langle\alpha|\beta\rangle = \qquad \boxed{\vec{u}\cdot\vec{v} = |\vec{u}|\cdot|\vec{v}|\cos\theta}$$

Cauchy-Schwartz inequality $\qquad \vec{v}\cdot\vec{u} \leq |\vec{u}||\vec{v}|$

$$\langle\alpha|\beta\rangle\langle\beta|\alpha\rangle \leq \langle\alpha|\alpha\rangle\langle\beta|\beta\rangle$$

$$\langle\beta|\alpha\rangle = \langle\alpha|\beta\rangle^*$$

$$|\langle\alpha|\beta\rangle|^2 \leq \langle\alpha|\alpha\rangle\langle\beta|\beta\rangle = \sigma_A^2\,\sigma_B^2$$

$$\langle\alpha|\beta\rangle = \langle\psi|(A-\langle A\rangle)^{+}(B-\langle B\rangle)|\psi\rangle$$

$$= \langle\psi|(AB - A\langle B\rangle - \langle A\rangle B + \langle A\rangle\langle B\rangle)|\psi\rangle$$

$$= \langle\psi|AB|\psi\rangle - \langle A\rangle\langle B\rangle - \langle A\rangle\langle B\rangle + \langle A\rangle\langle B\rangle$$

$$= \langle AB\rangle - \langle A\rangle\langle B\rangle$$

$$\langle\beta|\alpha\rangle = \langle\psi|(B-\langle B\rangle)^{+}(A-\langle A\rangle)|\psi\rangle$$

$= \langle \psi | BA - B\langle A \rangle - A\langle B \rangle + \langle B \rangle \langle A \rangle) | \psi \rangle$

$= \langle BA \rangle - \langle A \rangle \langle B \rangle - \langle A \rangle \langle B \rangle + \langle A \rangle \langle B \rangle$

$= \langle BA \rangle - \langle A \rangle \langle B \rangle$

Consider $z \in \mathbb{C}$

$|z|^2 = Re^2(z) + Im^2(z)$ ✓

$z = a + ib \qquad z^* = a - ib$

$|z|^2 = z z^* = (a+ib)(a-ib) = a^2 - iab + iab + b^2$

$= a^2 + b^2$

$|z|^2 = Re^2(z) + Im^2(z) \geq Im^2(z) \qquad Re^2(z) \geq 0$

$Im(z) = \frac{z - z^*}{2i} \qquad z = \langle \alpha | \beta \rangle$

$z^* = \langle \beta | \alpha \rangle$

$\frac{z - z^*}{2i} = \frac{1}{2i}\left(\langle \alpha | \beta \rangle - \langle \beta | \alpha \rangle \right)$

$= \frac{1}{2i}\left[\langle AB \rangle - \langle A \rangle \langle B \rangle - (\langle BA \rangle - \langle A \rangle \langle B \rangle) \right]$

$= \frac{1}{2i}\left(\langle AB \rangle - \langle A \rangle \langle B \rangle - \langle BA \rangle + \langle A \rangle \langle B \rangle \right)$

$= \frac{1}{2i}\left(\langle AB \rangle - \langle BA \rangle \right)$

$= \frac{1}{2i}\left(\langle AB - BA \rangle \right)$

$\langle AB \rangle = \langle \psi | AB | \psi \rangle$

$\langle AB - BA \rangle = \langle \psi | (AB - BA) | \psi \rangle$

$= \langle AB \rangle - \langle BA \rangle)$

$Im\left(\langle \alpha | \beta \rangle \right) = \frac{1}{2i} \langle [A, B] \rangle$

$Im^2\left(\langle \alpha | \beta \rangle \right) \leq | \langle \alpha | \beta \rangle |^2 \leq \langle \alpha | \alpha \rangle \langle \beta | \beta \rangle$

$\leq \sigma_A^2 \sigma_B^2$

$$\mathrm{Im}^2(\langle\alpha|\beta\rangle) \leq \sigma_A^2\sigma_B^2$$

$$\boxed{\left(\frac{1}{2i}\langle[A,B]\rangle\right)^2 \leq \sigma_A^2\sigma_B^2}$$

Generalized uncertainty principle, which applies to any pair of operators that do not commute.

$$[\hat{x},\hat{p}]f(x) \qquad \hat{p} \rightarrow -i\hbar\partial_x$$

$$\begin{aligned}
&= (\hat{x}\hat{p} - \hat{p}x)f(x)\\
&= -i\hbar x\partial_x f(x) - (-i\hbar)\partial_x xf(x)\\
&= -i\hbar x\partial_x f + i\hbar(f + x\partial_x f)\\
&= -i\hbar x\partial_x f + i\hbar f + i\hbar x\partial_x f\\
&= i\hbar f \Rightarrow
\end{aligned}$$

$$[\hat{x},\hat{p}] = i\hbar\mathbb{1}$$

$$\langle[x,p]\rangle = i\hbar$$

$$\left(\frac{1}{2i}\langle[x,p]\rangle\right)^2 = \left(\frac{1}{2i}i\hbar\right)^2 = \left(\frac{\hbar}{2}\right)^2 \leq \sigma_x^2\sigma_p^2$$

$$\boxed{\frac{\hbar}{2} \leq \sigma_x\sigma_p}$$